Counting with Theta Characteristics

Theta Characteristics on Algebraic Curves as a Tool for Solving Geometric Problems

by

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Contents

Chapter 1

Introduction

In this thesis, we will discuss the theory of theta characteristics of algebraic curves and how they allow us to count interesting geometric objects.

1.1 Structure

We will begin by introducing the mathematical background necessary to understand the theta characteristics and their main properties. This will include a brief discussion of sheaf theory, category theory, scheme theory, some ring theory, and a little bit of linear algebra. We will then proceed to introduce a few fundamental results, most without proof, that will be essential for the arguments in the main chapters of this thesis. As the theory presented in this chapter comes from many different directions, a multitude of sources was necessary. The main inputs for this background chapter came from [Har77], [Har92], [EH24], and [Vak25].

In the second chapter, we define theta characteristics on smooth projective curves and state a theorem relating their number and parity decomposition to the genus of the curve on which they live. The main sources for this chapter were [EH24] and [Dol12].

In the third chapter, we apply the theory developed in the previous chapter to solve a geometric problem. It turns out that the number of bitangents of a projective quartic plane curve stands in one-to-one correspondence with the odd theta characteristics of this curve. This will allow us to count these bitangents. In particular, we will see that this number is independent of the smooth quartic plane curve itself. The main sources for this chapter were [Dol12] and [EH24], but I also want to mention the manuscript [Kai] which proved very useful in getting a better understanding of the theory.

In the fourth chapter, we will extend the theory of theta characteristics to Gorenstein curves which include a wide range of singular curves in addition to the smooth curves we

discuss in chapter two. This generalization is due to Joe Harris whose notes [Har82] also provided the main source for this chapter.

Finally, the fifth chapter will be concerned with an application of the theory of theta characteristics on singular curves. In particular, we will see that the Apollonius problem, concerned with finding the tangent circles to a given triple of circles in the affine plane, can be solved using the theory developed in the previous chapter. [Har82] was my main inspiration and source for this chapter.

1.2 History

The idea of theta characteristics appeared for the first time in the 19th century when [Göp77] and [Ros46] used the theory of even and odd theta functions on Jacobi's inversion formula for the setting of genus 2. To understand the relations between theta constants Frobenius later developed an algebra of characteristics. He distinguished between two different kinds of characteristics: period and theta. This distinction and its translation into modern language turned out to be extremely useful in the context of transformation laws of theta functions and the symplectic group $\mathrm{Sp}(F_2^{2g})$'s action on characteristics. There was a lot of subsequent work on this field, but it needed Atiyah and Mumford to develop the modern theory of theta characteristics. The modern functorial approach allowed the theory to be extended to new areas such as singular curves which was done by Joe Harris [Har82]. Cornalba later constructed the moduli space of stable spin curves and Dolgachev and Ortland translated many older ideas about theta characteristics into modern language. A far more extensive historical discussion of theta characteristics can be found in [Far12].

Chapter 2

Background

In this chapter we will introduce most of the theoretical background necessary to understand the main ideas of this thesis. This will include a brief introduction to sheaf theory, some basic definitions and results from ring theory, some useful linear algebra facts, a discussion of divisors, some basic combinatorial results as well as a short discussion of the Weil pairing, Gorenstein rings and some elementary results about exact sequences.

For brief discussions about schemes and normalizations, see the appendix. For blow ups and cohomology theory we refer to [Vak25] for an in-depth introduction.

Throughout the thesis, we take the base field k to be $\mathbb{C}$, although most, if not all, of the results remain valid over an arbitrary algebraically closed field.

2.1 Curves

We will assume that the reader has a general familiarity with curves. Most of the content of this section on curves should not be new, but we recall it here as a quick reference point.

The most important distinction between curves in this thesis is based on whether they are singular or not which we define as follows.

Definition 2.1.1 (Singularity)**.** *Let C be a curve and p a point of C. We call p nonsingular or smooth if $\mathcal{O}_{C,p}$ is a regular local ring. If p is not smooth, we call it singular.*

We call C nonsingular or smooth if every point p of C is nonsingular. If C has a point p that is singular, we call C singular [Har77].

An important result that we will use many times in the proofs presented in later chapters is the following special case of Bézout's theorem.

Theorem 2.1.2 (Bézout's Theorem)**.** *Let $C \neq D$ be plane curves of degrees c, d. Then $C \cap D$ consists of $c \cdot d$ points (with multiplicity).*

Proof. See [Har77] Corollary 7.8 for a proof. □

Another useful result connects the degree and the genus of plane curves.

Lemma 2.1.3. *A smooth curve C of degree d has genus*

$$g = \frac{(d-1)(d-2)}{2}.$$

Proof. For a proof see page 167 in [Har92]. □

Also useful for genus computations is the following theorem.

Theorem 2.1.4 (Petri's Theorem)**.** *If C is the intersection of a quadric and a cubic surface, then it has genus 4.*

Proof. For a proof see [Vak25]. □

Our count of theta characteristics on a curve will rely on reducing the problem to counting them on so called hyperelliptic curves.

Definition 2.1.5 (Hyperelliptic Curve)**.** *We say that a curve C is hyperelliptic if it has genus at least 2 and there exists a finite morphism $f : C \to \mathbb{P}^1$ of degree 2 [Har77].*

The following two lemmas about hyperelliptic curves will turn out to be particularly helpful.

Lemma 2.1.6. *Smooth quartic plane curves are nonhyperelliptic.*

Proof. By lemma 2.1.3 a quartic plane curve has genus 3. For the remainder of the proof see [EH24] page 121. □

Lemma 2.1.7. *On a hyperelliptic curve C of genus $g \geqslant 2$ the degree 2 map to $\mathbb{P}^1$ is unique up to automorphisms of $\mathbb{P}^1$.*

Proof. For a proof see proposition 19.5.7 in [Vak25]. □

The following definition is not central to the discussions about theta characteristics, but the class of local intersection curves provides good examples for Gorenstein curves 5.0.1.

Definition 2.1.8 (Local Intersection Curve)**.** *If p is a point on a curve $C \subset \mathbb{P}^n$ with local dimension k, then we say that C is a local complete intersection at p if there exists an affine neighborhood of p in $\mathbb{P}^n$ such that the ideal $I(U \cap X)$ sitting inside the ring of regular functions $A(U)$ on U is generated by $n - k$ elements .*

If C is a local complete intersection for every point $p \in C$, then we call C a local complete intersection curve [Har92].

Intuitively, a local complete intersection of dimension k embedded in an n dimensional space is locally obtained by imposing $n - k$ independent conditions.

Most of the curves that come to mind on first thought are local intersection curves, but here is a less standard example.

Example 2.1.9. *A discrete set of points in $\mathbb{P}^2$. Here $n = 2$ and $k = 0$. In local coordinates each point is described by the equations $x = 0$ and $y = 0$. Note that this example evidently is not a global complete intersection curve.*

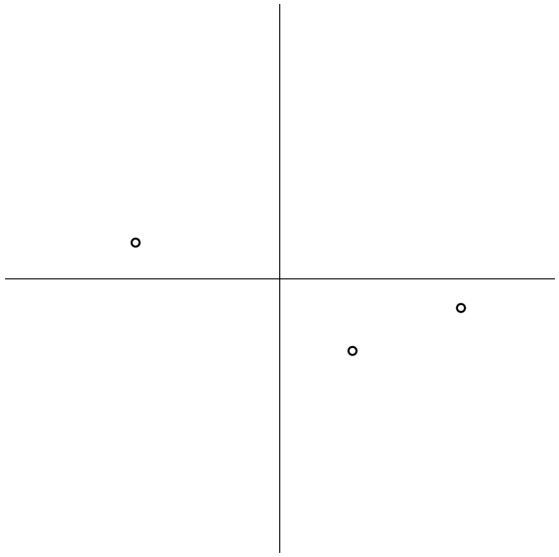


Figure 2.1: Affine chart of the local complete intersection curve given by three points in $\mathbb{P}^2$.

We finish this section with a collection of different curve facts that we will use throughout the thesis. The least elementary result is the following theorem that we will use to reduce the problem of counting theta characteristics on smooth curves to the hyperelliptic case.

Theorem 2.1.10. *Let M_g be the space of curves of genus g. Then M_g is irreducible as a variety. In particular, it is connected.*

Proof. For a proof see [DM69] page 93. □

2.2 Sheaf Theory

Sheaf Theory gives us the language to talk about line bundles on a curve and thus allow us to define theta characteristics. It is the primary language that we will use in this thesis. This introduction uses [Har77] as its main source and additional sources where indicated.

Before we can define what sheaves are we have to introduce the more general notion of a presheaf.

Definition 2.2.1 (Presheaf)**.** *If X is a topological space, then we say that a presheaf $\mathcal{F}$ of abelian groups on X is the following data*

(I) To every $U \underset{open}{\subseteq} X$ there is an assignment of an abelian group $\mathcal{F}(U)$.

(II) For every pair of open sets U, V of X such that $V \subseteq U$ there is a "restriction" morphism of abelian groups
$\rho_V^U : \mathcal{F}(U) \to \mathcal{F}(V)$.

such that

(i) The empty set gets assigned the trivial group: $\mathcal{F}(\varnothing) = \{0\}$.

(ii) For every $U \underset{open}{\subseteq} X$ the respective morphism of abelian groups $\rho_U^U : \mathcal{F}(U) \to \mathcal{F}(U)$ is the identity map.

(iii) The restriction morphism factors through intermediate open subsets: If $W \subseteq V \subseteq U$, then $\rho_W^U = \rho_W^V \circ \rho_V^U$.

It is not difficult to check that all of the following examples really satisfy the definition of a presheaf.

Example 2.2.2 (Presheaf)**.** *The constant pre-sheaf: Assign to every open set in the topology the same abelian group A.*

We will see more interesting examples after we restrict ourselves to the additional restrictions of a sheaf.

Definition 2.2.3 (Sheaf)**.** *We call a presheaf $\mathcal{F}$ on a space X a sheaf if it additionally satisfies that*

(i) (Locality) for all open sets $U \subseteq X$, open coverings $\{V_i\}$ of U, and $s \in \mathcal{F}(U)$ the condition $s\,|_{V_i} = 0$ for all i implies that $s = 0$.

(ii) (Gluing) for all $U \underset{open}{\subseteq} X$, open coverings $\{V_i\}$ of U, and $s_i \in \mathcal{F}(V_i), s_j \in \mathcal{F}(V_j)$ where s_i and s_j agree on $V_i \cap V_j$, there exists an element $s \in \mathcal{F}(U)$ such that $s\,|_{V_i} = s_i$ for all i.

Note that the locality condition implies that s is unique. An equivalent and more elegant definition of a sheaf is the following.

Definition 2.2.4 (Sheaf Definition through Exact Sequence)**.** *A sheaf is a presheaf that makes the following sequence exact*

$$0 \longrightarrow F(U) \overset{f \mapsto f|_{U_i}}{\longrightarrow} \prod_{i \in I} F(U_i) \overset{(f_i) \mapsto (f_i - f_j)|_{U_{ij}}}{\longrightarrow} \prod_{(i,j) \in I} F(U_{ij}).$$

Intuitively, sheaves encode local structural data about the underlying space through the groups they assign locally.

In the context of this thesis, one of the most important examples is the sheaf of regular functions on a variety.

Example 2.2.5 (Sheaf of Regular Functions)**.** *Let X be a variety. Then we write $\mathcal{O}_X$ for the sheaf of regular functions:*

$$\mathcal{O}_X(U) := \{f \mid f \text{ is regular on } U\}.$$

Recall that we call a function regular if it can locally be written as a quotient of polynomials where the denominator is locally nonvanishing.

The most important property of a sheaf compared to a general presheaf is that we get a local group for any point of the underlying topological space.

Definition 2.2.6 (Stalk)**.** *Let X be a topological space and $\mathcal{F}$ a presheaf on X. For each point $p \in X$ we define the stalk $\mathcal{F}_p$ as the direct limit [1] of the groups $\mathcal{F}(U)$ for all open neighborhoods U of p in X, built from the restriction morphisms ρ.*

$$\mathcal{F}_p = \varinjlim_{p \in U} F(U)$$

[1] For a good source talking about direct limits see [Vak25] chapter 1.

In other words, elements of $\mathcal{F}_p$ are represented by tuples (U, s) of open neighborhoods U of p and sections $s \in \mathcal{F}(U)$. We identify two such tuples (U, s) and (V, t) if there exists an open neighborhood W of p contained in $U \cap V$ on which the restrictions of s and t agree. Equivalently, the stalk consists of the germs of sections of $\mathcal{F}$ at p.

Sometimes we will write, $\mathrm{Sec}(\mathcal{F}_p)$ for the group of sections of $\mathcal{F}_p$ to emphasize that we are talking about sections..

Our previous example of a sheaf also provides us with an illustration of the concept of a stalk.

Example 2.2.7 (Stalk)**.** *Let C be a curve and $\mathcal{O}_C$ be its sheaf of regular functions. Then the stalk $\mathcal{O}_{C,p}$ at any point $p \in C$ is the local ring of p on C.*

Now that we know what objects sheaves are we can move on to defining maps between them.

Definition 2.2.8 (Morphism of Presheaves)**.** *Let X be a topological space and let F and G be presheaves on X. Then we define a morphism of presheaves $\phi : F \to G$ to be a morphism of abelian groups on every open set $U \subseteq X$ satisfying commutativity in the diagram*

$$\begin{array}{ccc} F(U) & \xrightarrow{\phi(U)} & G(U) \\ \Big\downarrow{\scriptstyle \rho_V^U} & & \Big\downarrow{\scriptstyle \rho_V'^U} \\ F(V) & \xrightarrow{\phi(V)} & F(V) \end{array}$$

for $V \subseteq U$.

Definition 2.2.9 (Isomorphism of Presheaves)**.** *An isomorphism of presheaves is a morphism that has a left and a right inverse.*

Throughout our discussion of theta characteristics we will also need some vocabulary to denote certain classes of sheaves.

Definition 2.2.10 (Coherent Sheaf)**.** *A coherent sheaf $\mathcal{L}$ on a curve C is defined as the following data.*

1. *An affine open cover $\{U_i\}_{i \in I}$ of C,*
2. *$\forall i \in I$, a finitely generated $\mathcal{O}_C(U_i)$ module L_i, and*

3. $\forall i, j \in I$*, an isomorphism*

$$\sigma_{ij} : L_i \mid_{U_i \cap U_j} \to L_j \mid_{U_i \cap U_j}$$

such that for all $k \in I$

$$\sigma_{jk}\sigma_{ij} = \sigma_{ik}$$

[EH24].

Definition 2.2.11 (Invertible Sheaf)**.** *A coherent sheaf* $\mathcal{L}$ *on* C *is said to be invertible if*

$$L \mid_U \cong \mathcal{O}_U$$

for all U *in some open cover* $\{U_i\}_{i \in I}$ *of* C*.*

It turns out that invertible sheaves of a curve C form a group with the tensor product as its group operation. The identity is the structure sheaf. The inverse of an invertible sheaf in this group is its dualizing sheaf. We only need the notion of a dualizing sheaf in the context of a curve where it is defined as follows.

Definition 2.2.12 (Dualizing Sheaf)**.** *Let* C *be a curve over a field* k*. A dualizing sheaf for* C *is the data of*

1. a coherent sheaf ω_C° *on* C *and*

2. a morphism $t : H^n(C, \omega_C^\circ) \to k$

such that for all coherent sheaves $\mathcal{F}$ *on* C *the composition* $t \circ \phi$ *gives an isomorphism*

$$\mathit{Hom}(\mathcal{F}, \omega_C^\circ) \xrightarrow{\cong} H^n(C, \mathcal{F})^*$$

where ϕ *denotes the natural pairing*

$$\mathit{Hom}(\mathcal{F}, \omega_C^\circ) \times H^n(C, \mathcal{F}) \to H^n(C, \omega_C^\circ)$$

[Har77].

Lemma 2.2.13 (Degree of Dualizing Sheaf)**.** *The degree of the dualizing sheaf* ω_C *of a proper curve* C *is* $2g - 2$ *where* g *is the genus of the curve.*

Proof. For a proof see [Sta26] lemma 53.8.3. □

Definition 2.2.14 (Structure Sheaf)**.** *We define* $\mathcal{O}_{\operatorname{Spec} A}(D(f))$ *to be the localization of* A *at the multiplicative set functions which do not vanish outside the zero set of* f *and call it the structure sheaf of* $\operatorname{Spec} A$*.*[2]

[2]For a short discussion about spectra see the appendix at 7.1.1.

The notion of a dualizing sheaf allows us to define the so called canonical sheaf of a curve which will play a crucial role for defining theta characteristics.

Definition 2.2.15 (Canonical Sheaf)**.** *Let C be a reduced curve 7.1.7 (not necessarily smooth). Then the canonical sheaf of C is its dualizing sheaf ω_C [Har82].*

If the curve C is actually smooth, we get a more intuitive definition of the canonical sheaf.

Definition 2.2.16 (Canonical Sheaf)**.** *Let C be a nonsingular curve and let Ω_C be the sheaf of regular 1-forms on C. Then we call*

$$\omega_C := \Omega_C$$

the canonical sheaf of C [Har77].

Examples of curves with canonical invertible sheaves are all nonsingular curves or curves on nonsingular surfaces.

2.3 Divisors

Another important topic for our discussion of theta characteristics is the theory of divisors which live in close relationship with their associated sheaves.

We start with the definition of a divisor on a smooth curve as it is much more intuitive in this context. Later we will extend the definition to singular curves as well.

Definition 2.3.1 (Divisor)**.** *Let C be a smooth projective curve. A divisor on C is a finite formal integral sum of points on C.*

In other words, a divisor is a formal sum of the form

$$\sum_{p \in C} m_p \cdot p$$

where all but finitely many of the m_p are zero. We call m_p the multiplicity of the point p in the divisor D.

Definition 2.3.2 (Principal Divisor)**.** *Let C be a smooth curve and $U \subset C$ an open subset. Then we define the divisor (f) associated to a not identically zero regular function $f \in \mathcal{O}_C(U)$ to be*

$$(f) := \sum_{p \in U} \mathrm{ord}_p(f) \cdot p.$$

We can similarly associate a divisor to any rational function $h = f/g$ by defining

$$(h) := (f) - (g).$$

The divisors of this form are called principal divisors [EH24].

The notion of principal divisors now allows us to define an equivalence relation on the set of all divisors.

Definition 2.3.3 (Linear Equivalence)**.** *We say that two divisors D and E are linearly equivalent if there exists some principal divisor (h) such that*

$$D - E = (h).$$

This equivalence relation allows us to think about divisors in equivalence classes.

Definition 2.3.4 (Picard Group)**.** *The divisors of a curve C form a free abelian group $\operatorname{Div} C$ generated by the points of C. Taking the quotient by the group of principal divisors $\operatorname{Div}_0 C$ we get the Picard group of linear equivalence classes denoted by $\operatorname{Pic} C$.*

$$\operatorname{Pic} C := \operatorname{Div} C / \operatorname{Div}_0 C.$$

We can also define a partial order on $\operatorname{Div} C$ by writing $E \geqslant D$ for $D = \sum m_p \cdot p$ and $E = \sum n_p \cdot p$ if $n_p \geqslant m_p$ for all p.

Definition 2.3.5 (Effective Divisor)**.** *We call a divisor effective if all of the multiplicities m_p are nonnegative.*

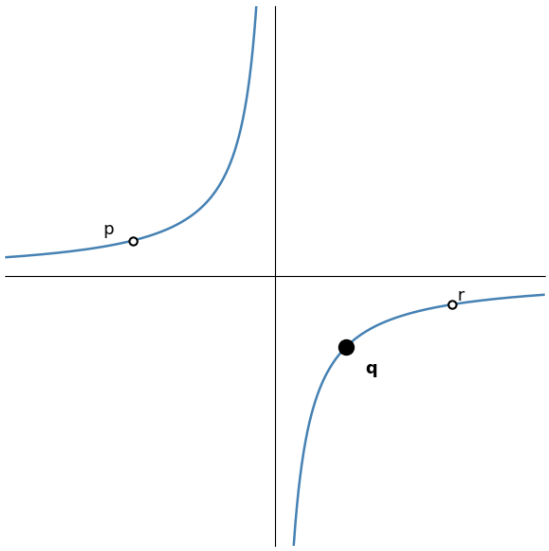


Figure 2.2: A visualization of the effective divisor $D = p + 2q + r$ on the real affine chart of the curve $X_0X_2 + X_1X_3 = 0$. It has degree 4.

Definition 2.3.6 (Degree of a Divisor)**.** *Let $D = \sum_{p \in C} m_p \cdot p$. Then we define the degree of D to be*

$$\deg D = \sum m_p.$$

On singular curves the definition of a divisor is less intuitive. We call them *Cartier divisors*. To define them we first have to define the *effective Cartier divisors* that generate them.

Definition 2.3.7 (Effective Cartier Divisor)**.** *Let C be a possibly singular curve and $\{U_i\}_{i \in I}$ an open cover of C. An effective Cartier divisor of C is a subscheme that is locally defined on the U_i as the zero set of one nonzero principal divisor corresponding to some $f \in \mathcal{O}_C(U_i)$ [EH24].*

Now we are ready to introduce particular divisors that we will need for our theta characteristics discussion. Let us start with the important notion of a canonical divisor.

Definition 2.3.8 (Canonical Divisor)**.** *We call a divisor in the linear series corresponding to ω_C a canonical divisor and denote it by K_C.*

Lemma 2.3.9. *Let C be a curve of genus g, then* $\deg K_C = 2g - 2$.

Proof. A proof can be found in [Sta26] Lemma 53.8.4. □

The three deepest results that we will use about divisors are the following. For this result and for later chapters we will use the notation

$$h^0(-) := \dim H^0(-).$$

For divisors we will abuse notation and write $h^0(D)$ for $h^0(\mathcal{O}(D))$.

Theorem 2.3.10 (Riemann-Roch Theorem)**.** *Let C be a smooth curve of genus g and D be a divisor. Then*

$$h^0(D) - h^0(K - D) = \deg D - g + 1$$

where K is the canonical divisor of C.

Proof. For a proof see [GH94] page 246. □

Definition 2.3.11 (Special Divisor)**.** *We call a divisor special if $K_C - D$ is linearly equivalent to some effective divisor [Har77].*

Theorem 2.3.12 (Clifford's Theorem)**.** *If D is an effective special divisor on a curve C, then*

$$\dim |D| \leqslant \frac{1}{2} \deg D.$$

We get equality if and only if either $D = 0$, $D = K$ or C is hyperelliptic and D is a multiple of the unique g^1_2 on C.

Proof. For a proof see [Har77] page 344. □

Theorem 2.3.13 (Adjunction Formula)**.** *Let X be a smooth projective variety and $Y \hookrightarrow X$ a smooth codimension 1 subvariety. Then*

$$K_Y = (K_X + Y)\,|_Y$$

where we write K_Z for the canonical divisor class on Z.

Proof. This follows from proposition 29.4.8 in [Vak25]. □

One of the most fundamental notions in our later discussions is the sheaf associated to a divisor. We again begin the smooth case, since it is more intuitive before stating the definition for singular schemes.

Definition 2.3.14 ($\mathcal{O}_D(D)$)**.** *Let C be a smooth projective curve and $D = \sum m_p \cdot p$ a divisor on C. Then we define the sheaf $\mathcal{O}(D)$ associated to D to be the sheaf whose sections on open sets are given by*

$$\mathcal{O}_C(D)(U) := \{\textit{rational functions } f \mid \operatorname{ord}_p(f) + m_p \geqslant 0 \textit{ for all } p \in U\}.$$

In other words, at any $p \in C$ the sections are exactly the rational functions with a pole of order at most m_p. A negative m_p means that f must be regular at p and have a zero of order at least $-m_p$ there [EH24].

In the context of singular schemes we define the associated sheaf as follows.

Definition 2.3.15 ($\mathcal{O}_D(D)$ for singular schemes)**.** *Let X be a scheme and let $D = E - F$ be a general (Cartier) divisor. Then we define*

$$\mathcal{O}_X(D) := \mathcal{O}_X(E) \otimes \mathcal{O}_X(F)^{-1}$$

[EH24].

We finish the section on divisors with some notation that will turn out to be useful.

Definition 2.3.16 (Linear Series)**.** *Let C be a curve. Then a linear series on C is a pair $\mathcal{V} = (\mathcal{F}, V)$ of an invertible sheaf $\mathcal{F}$ of degree d on C and a nonzero vector space V of global sections of $\mathcal{F}$. The dimension of $\mathcal{V}$ is defined by the projective dimension of V:*

$$\dim \mathcal{V} := \dim_{\mathbb{C}} V - 1.$$

We write g^r_d to denote a linear series of degree d and dimension r [EH24].

Intuitively, a linear series is a family of divisors that move linearly on the curve. Consider for example $\mathcal{V} = (\mathcal{F}, V)$ where $V \subseteq H^0(C, \mathcal{F})$. Then each nonzero section $s \in V$ has a divisor defined by the zeros of s. As s varies in V, these divisors vary in a projective family which we call the linear series.

Example 2.3.17.

Definition 2.3.18 (Pencil)**.** *We call a linear series of dimension 1 a pencil [EH24].*

Historically the notion of a pencil referred to the lines passing through a point.

2.4 Ring Theory

While we assume general knowledge of modern algebra throughout this thesis, we recall some basic definitions from ring theory as a quick reference.

We begin by introducing the notion of a multiplicative subset of a ring which will allow us to define afterwards the localization of a ring.

Definition 2.4.1 (Multiplicative Subset of a Ring)**.** *Let S be a subset of a ring R. If S contains the 1-element of R and is closed under multiplication, we call it a multiplicative subset of R [Sta26].*

Definition 2.4.2 (Localization of a Ring)**.** *Let R be a ring and S a multiplicative subset of R. Then we define $S^{-1}R$ to be the set of equivalence classes of the equivalence relation on $R \times S$ defined by*

$$(x, s) \sim (y, t) \iff \exists u \in S \text{ such that } (xt - ys)u = 0.$$

We use $\frac{x}{s}$ to denote the equivalence class of the element $(x, s) \in R \times S$. We can give $S^{-1}R$ the structure of a ring by defining

$$\frac{x}{s} + \frac{y}{t} = \frac{xt + ys}{st}, \qquad \frac{x}{s} \cdot \frac{y}{t} = \frac{xy}{st}.$$

We call this ring $S^{-1}R$ the localization of R [Sta26].

A less well-known class of rings are the Gorenstein rings which we will need to define the Gorenstein curves that underlie our discussion of theta characteristics on singular curves.

Definition 2.4.3 (Gorenstein Ring)**.**

1. *If A is a Noetherian local ring and $A[0]$ is a dualizing complex for A, then we say that A is a Gorenstein local ring.*

2. *If A is a Noetherian ring and A_p is a Gorenstein local ring for every prime p in A, then we call A a Gorenstein ring.*

2.5 Linear Algebra

We assume that the reader is familiar with elementary linear algebra. In this section we will only introduce the notion of an isotropic subspace of a vector space and some of their properties which will turn out to be very useful when counting theta characteristics on smooth curves.

Definition 2.5.1 (Isotropic Subspace)**.** *Let V be an even-dimensional vector space over the complex numbers equipped with a nondegenerate bilinear form Q and let $\Lambda \subset V$ be a subspace which satisfies $Q(\Lambda, \Lambda) = 0$. Then we call Λ an isotropic subspace of V [EH24].*

Lemma 2.5.2 (Properties of Isotropic Subspaces)**.** *Let V be a $2n$ dimensional complex vector space and Q be its nondegenerate bilinear form.*

1. *The maximal isotropic subspaces of V with respect to Q are n-dimensional.*
2. *The maximal isotropic subspaces of V with respect to Q form a $\binom{n}{2}$-dimensional subvariety of the Grassmannian $G(n, V)$. It consists of exactly 2 connected components.*
3. *For any two maximal isotropic subspaces Λ, Λ' of V the following two statements are equivalent:*
   (a) $\dim(\Lambda \cap \Lambda') \equiv n (\textit{mod}\ 2)$.
   (b) *Λ and Λ' belong to the same connected component of the subvariety of maximal isotropic subspaces.*

Proof. For a proof see [GH94], pp. 735-740. □

2.6 Quadratic Forms & Weil Pairing

Let f be a rational function on C and let D be a divisor on C such that the support of D and the zeros and poles of f do not intersect. Then define

$$f(D) := \prod_{p \in C} f(p)^{\text{mult}_p D}.$$

It can be quite easily checked that this definition satisfies the *Weil Reciprocity Law* $f((g)) = g((f))$. For $C = \mathbb{P}^1$ this follows immediately from the definition and some minor alge-

braic manipulations. For any other C, this follows from pushing f forward along $C \to \mathbb{P}^1$ and constructing g by pushing down along the same map by taking the norm.

Let us introduce some notation.

Definition 2.6.1 (Jacobian)**.** *Throughout this thesis, we write $J(C)$ for the Jacobian of a curve C and $J_2(C)$ for its 2-torsion subgroup. For our purposes, the Jacobian is identified with the group of divisor classes of degree 0.*

Now we can define the Weil pairing.

Definition 2.6.2 (Weil Pairing)**.** *The Weil pairing is a bilinear form*

$$\lambda : J_2 \times J_2 \to \mathbb{Z}/2$$

defined by

$$\lambda(D, E) = \frac{1}{\pi i} \log \frac{f(E)}{g(D)}$$

[Har82].

2.7 Exact Sequences

Lemma 2.7.1. *If we have an exact sequence of the form*

$$A \to B \to C \to D \to 0$$

then

$$0 \to B/\operatorname{Im} A \to C \to D \to 0$$

is a short exact sequence.

Proof. This follows immediately from the exactness of the first sequence. The part $C \to D \to 0$ remains unchanged and we have that $\operatorname{Im}(B \to C) = \operatorname{Im}(B/\operatorname{Im} A \to C)$ because $\ker(B \to C) = \operatorname{Im}(A \to C)$. By the same token we also have that $\ker(B/\operatorname{Im} A) = 0$ which finishes the exactness proof for the second sequence. □

Lemma 2.7.2 (Snake Lemma)**.** *Suppose the diagram of abelian groups with exact rows below is commutative.*

$$\begin{array}{ccccccccc} & & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \gamma} & & \\ 0 & \longrightarrow & E & \longrightarrow & F & \longrightarrow & G & & \end{array}$$

Then there is an exact sequence

$$\ker \alpha \to \ker \beta \to \ker \gamma \to \operatorname{coker} \alpha \to \operatorname{coker} \beta \to \operatorname{coker} \gamma.$$

If $A \to B$ is injective, then the first map of this new exact sequence is injective.

Proof. See [Sta26] 10.4 Snake lemma for a proof. □

2.8 Miscellaneous

Here we present two lemmas from combinatorics and complex analysis that each do not deserve their own subsection but are nonetheless important for the proofs in subsequent chapters.

Lemma 2.8.1 (Counting Lemmas)**.** *For any $n \in \mathbb{N}^{>0}$, the following combinatorial identities hold true.*

1.

$$\sum_{k=0}^{n} \binom{2n}{2k} = \sum_{k=0}^{n-1} \binom{2n}{2k+1} = 2^{2n-1},$$

2.

$$\sum_{k=0}^{n} \binom{4n}{4k} = 2^{4n-2} + (-1)^n 2^{2n-1} \quad \textit{and} \quad \sum_{k=0}^{n-1} \binom{4n}{4k+2} = 2^{4n-2} - (-1)^n 2^{2n-1},$$

3.

$$\sum_{k=0}^{n} \binom{4n+2}{4k+1} = 2^{4n} + (-1)^n 2^{2n}, \quad \textit{and} \quad \sum_{k=0}^{n-1} \binom{4n+2}{4k+3} = 2^{4n} - (-1)^n 2^{2n}$$

[EH24].

Proof. For the sake of readability we have moved the proof of this lemma to the appendix 7.2. □

Theorem 2.8.2 (Global Sum of Residues)**.** *The global sum of residues of a rational function on a curve C is zero.*

Proof. For a proof see [For81] page 80, theorem 10.21. □

Chapter 3

Counting Theta Characteristics on Smooth Curves

We will first introduce the theory of theta characteristics on smooth curves. In this chapter we will first define theta characteristics, then state the main theorem which provides a count of the theta characteristics given a smooth curve. Finally we will introduce some intermediate results necessary for proving the counting theorem and use them to prove it. The main sources for this chapter were [EH24] and [Dol12].

Definition 3.0.1 (Theta Characteristic)**.** *Let C be a smooth curve. We call an isomorphism class of line bundles $\mathcal{L}$ on C a theta characteristic on C if its tensor square is equal to the canonical line bundle ω_C of C, i.e.*

$$\mathcal{L}^{\otimes 2} \cong \omega_C.$$

We say that $\mathcal{L}$ is an even theta characteristic if $h^0(\mathcal{L}) := \dim H^0(\mathcal{L})$ is even and say it is odd if $h^0(\mathcal{L})$ is odd.

For examples of theta characteristics see the next chapter.

Now let us dive right into it and state the main result of this chapter providing a count of the theta characteristics and how they split into even and odd ones.

Theorem 3.0.2 (Number of even & odd theta characteristics)**.** *Let C be a smooth curve with genus g. Then there are 2^{2g} theta characteristics on C of which $2^{g-1}(2^g + 1)$ are even and $2^{g-1}(2^g - 1)$ are odd.*

This theorem immediately implies that, for example, curves of genus 3 have $2^6 = 64$ theta characteristics of which $2^2(2^3 + 1) = 36$ are even and $2^2(2^3 - 1) = 28$ are odd.

Before we can prove this theorem, we will have to introduce some machinery beginning with the following result.

Theorem 3.0.3. *Consider a family $\mathcal{C} \to B$ of smooth curves and an invertible sheaf $\mathcal{L}$ on $\mathcal{C}$ such that $(\mathcal{L} \mid_{C_b})^2 \cong K_{C_b}$ for each $b \in B$. Let $f : B \to \mathbb{Z}/2$ be defined by*

$$f(b) = h^0(\mathcal{L} \mid_{C_b}) \pmod 2.$$

Then f is locally constant.

Equivalently, we may interpret $\mathcal{L}$ as a family $\mathcal{L}_b$ of theta characteristics on the curves in the family of smooth curves.

Proof. To prove this theorem, we will translate it into a linear algebra problem to apply lemma 2.5.2. We will look at $H^0(\mathcal{L}(D)/\mathcal{L}(-D))$ as a vector space and use that $H^0(\mathcal{L})$ can be written as the intersection of two isotropic subspaces of that vector space. This will allow us to reduce the proof of the theorem to finding the dimension of this intersection which we can do by using lemma 2.5.2.

Suppose C is a smooth curve of genus g and $\mathcal{L}$ a theta characteristic on C. Let $p_1, \ldots, p_n$ be a set of distinct points of C where $n \geqslant g$ and define the divisor $D := p_1 + \cdots + p_n$. Define the vector space

$$V := H^0(\mathcal{L}(D)/\mathcal{L}(-D)).$$

Note that locally for an open set U that does not contain any of the p_i we have $\mathcal{O}_C(D) \mid_U = \mathcal{O}_C \mid_U$ and thus

$$\begin{aligned} \mathcal{L}(D) \mid_U &= \mathcal{L} \otimes \mathcal{O}_C(D) \mid_U \\ &= \mathcal{L} \mid_U \\ &= \mathcal{L} \otimes \mathcal{O}(-D) \mid_U \\ &= \mathcal{L}(-D) \mid_U . \end{aligned}$$

Therefore, away from the p_i we have that $\mathcal{L}(D)/\mathcal{L}(-D) = \mathcal{L}/\mathcal{L}$, the trivial sheaf. This means that the quotient sheaf can only be supported at the p_i. Near the p_i we have that $\mathcal{L}(D) = \mathcal{L} \otimes \mathcal{O}_C(D)$. Thus the sections of $\mathcal{L}(D)$ are exactly the sections of $\mathcal{L}$ that have poles of order $\leqslant 1$ at the points p_i. Similarly, we see that the sections of $\mathcal{L}(-D)$ vanish on all p_i. In particular, consider a local coordinate z centered at p_i. Then locally $\mathcal{L}(D) = z^{-1}\mathcal{O}_C$ and $\mathcal{L}(-D) = z\mathcal{O}_C$. Thus,

$$(\mathcal{L}(D)/\mathcal{L}(-D))_{p_i} = z^{-1}\mathcal{O}_C/z\mathcal{O}_C.$$

which is spanned by $\{z^{-1}, z\}$ and consequently each stalk has dimension 2. Now we can write

$$V \cong \bigoplus_{i=1}^{n} (\mathcal{L}(D)/\mathcal{L}(-D))_{p_i}$$

which implies that $\dim V = 2n$.

Define a bilinear form Q on V by

$$Q(\sigma, \tau) := \sum_i \mathrm{Res}_{p_i}(\sigma\tau).$$

Note that for simplicity of notation we wrote $\sigma\tau$ instead of its associated rational differential under the map $\mathcal{L}^{\otimes 2} \cong K_C$.

This bilinear form will allow us to translate the problem into a linear algebra problem. First, we define two subspaces Λ, Λ' of V by

$$\Lambda := H^0\left((\mathcal{L}/\mathcal{L}(-D)\right), \qquad \Lambda' := \mathrm{im}\left(H^0(\phi)\right)$$

where $\phi : H^0\left(\mathcal{L}(D)\right) \twoheadrightarrow H^0\left(\mathcal{L}(D)/\mathcal{L}(-D)\right)$ is the map induced by the quotient map of sheaves. Note that Λ is a subspace of V because the sections on $\mathcal{L}$ are holomorphic and the sections on $\mathcal{L}(D)$ are also allowed to have simple poles at D which gives us a natural inclusion of sheaves. We claim that both of the subspaces are isotropic subspaces of V with respect to Q. To see that Λ is isotropic, note that the product of any two of its elements have an associated regular differential which has no residues. Now for Λ' note that $H^0(\mathcal{L}(-D)) = 0$ which implies that ϕ is injective. The Riemann-Roch theorem 2.3.10 further tells us that $h^0\left(\mathcal{L}(D)\right) = n$. Thus Λ' is an n-dimensional subspace of V and it is isotropic since the sum of residues of a global rational differential on C is 0.

But now we have

$$H^0(\mathcal{L}) \cong \Lambda \cap \Lambda'$$

because Λ consists of the classes in $H^0(\mathcal{L}(D)/\mathcal{L}(-D))$ represented by sections of $\mathcal{L}$, whereas Λ' consists of the classes coming from global sections of $\mathcal{L}(D)$. Thus their intersection is exactly the image of $H^0(\mathcal{L})$.

Therefore $h^0(\mathcal{L}) = \dim(\Lambda \cap \Lambda')$ and the theorem follows from lemma 2.5.2. □

This theorem will be useful for proving the main theorem of this chapter. It allows us to reduce it to the case of hyperelliptic curves.

Lemma 3.0.4. *If C is a hyperelliptic curve and $p_1, \ldots, p_{2g+2} \in C$ are the ramification points of the unique degree 2 map $C \to \mathbb{P}^1$, then for any partition of $\{1, \ldots, 2g+2\}$ into two sets A, B of cardinality $g+1$, then*

$$\sum_{i \in A} p_i \sim \sum_{i \in B} p_i.$$

First we need a lemma showing that for a hyperelliptic curve the divisor class for any half of the ramification points is the same as the divisor class of the other half, and that there are no smaller relations.

Lemma 3.0.5. *Let C be a hyperelliptic curve of genus g expressed as a 2-sheeted cover of $\mathbb{P}^1$ with ramification points $p_1, \ldots, p_{2g+2}$. Let I_1, I_2 be subsets of $\{1, \ldots, 2g+2\}$ and*

$$D_i = \sum_{j \in I_i} p_j.$$

Then the divisors D_1, D_2 are linearly equivalent if and only if $I_1 = I_2$ or they both have cardinality $g+1$ and

$$I_1 \cup I_2 = \{1, \ldots, 2g+2\}.$$

Proof. The "if" direction follows immediately from lemma 3.0.4.

For the "only if" direction, we can assume that $I_1 \cap I_2 = \varnothing$ because we can just subtract away the common points of D_1 and D_2 since $D_1' + P \sim D_2' + P$ if and only if $D_1' \sim D_2'$. Linear equivalence preserves degree and thus

$$D_1 \sim D_2 \Rightarrow \deg D_1 = \deg D_2.$$

Denote their common degree by d. By assumption of the lemma we have $d \leqslant g+1$. We want to show that either $d = 0$ or $d = g+1$. Note that

$$D_1 \sim D_2 \iff D_2 - D_1 \equiv 0 \iff D_2 + D_1 \equiv 2D_1.$$

Next we claim that

$$d \leqslant g \Rightarrow \dim |2D_1| = d.$$

where $\dim |2D_1| = h^0(C, \mathcal{O}_C(2D_1)) - 1$. For $d < g$ this is a consequence of Clifford's theorem 2.3.12 which says that because C is hyperelliptic and $2D_1$ is a multiple [1]of the unique g^1_2 we have

$$\begin{aligned} \dim |2D_1| &= \frac{1}{2} \deg 2D_1 \\ &= \deg D_1 \\ &= d. \end{aligned}$$

For $d = g$ this is a consequence of the Riemann-Roch formula 2.3.10 which tells us that

$$h^0(2D_1) - h^0(K_C - 2D_1) = \deg 2D_1 - g + 1 = 2g - g + 1 = g + 1.$$

But now

$$\deg(K_C - 2D_1) = \deg K_C - \deg 2D_1 = (2g-2) - 2g = -2$$

and a divisor of negative degree cannot be effective. Thus it has no nonzero global sections and we get

$$h^0(2D_1) = g + 1 = d + 1.$$

This means that if $d \leqslant g$, then every divisor in $|2D_1|$ is a sum of d fibers of the 2 to 1 map of $C \to \mathbb{P}^1$. For any such divisor that is a sum of distinct points p_i the degree d must be 0 since any nonzero sum of fibers of the hyperelliptic map necessarily introduces multiplicities or conjugate pairs, and hence cannot consist of distinct simple points. $\square$

Finally, we can proceed to the proof of theorem 3.0.2.

Proof. We will show that the number of odd and even theta characteristics and thus also the total number of theta characteristics is independent of the choice of a smooth curve C of genus g. This will allow us to reduce the problem of counting theta characteristics for an arbitrary smooth curve C of genus g to counting them for a hyperelliptic curve of the same genus where the count can be carried out explicitly.

Let $\mathcal{M}_g$ be the moduli space of smooth genus g curves. Define the incidence variety

$$\mathcal{S}_g := \{(\mathcal{L}, C) \mid C \in \mathcal{M}_g,\ \mathcal{L}^{\otimes 2} \cong K_C\}$$

and define the projection maps

$$\pi : \mathcal{S}_g \to \mathcal{M}_g.$$

Note that for any $C \in \mathcal{M}_g$ the fiber of π over C consists exactly of the theta characteristics on C. We want to show that the number of elements in the fiber is independent of C which will allow us to reduce the count of theta characteristics on an arbitrary smooth curve to the count on hyperelliptic curves.

By [Nag90] the incidence variety $\mathcal{S}_g$ has exactly two connected components, one corresponding to the even and the other corresponding to the odd theta characteristics. Since π is a covering map, this means that there are only two possible fiber sizes, each corresponding to odd and even theta characteristics respectively. Therefore, the number of even theta characteristics is constant among curves of genus g and so is the number of odd ones. As a consequence, we can focus on counting theta characteristics on a hyperelliptic curve of genus g and this will tell us the number of theta characteristics on any smooth genus g curve.

To count the theta characteristics on a hyperelliptic curve, we will use lemma 3.0.5. Let C be a hyperelliptic curve of genus g expressed as a 2-sheeted cover of $\mathbb{P}^1$ with ramification points $p_1, \ldots, p_{2g+2}$.

Denote the class of the unique[2] g^1_2 on C by E, and assume D is the divisor corresponding to a theta characteristic $\mathcal{O}(D)$. Then $D + E$ will be effective because

$$\deg(D + E) = g - 1 + 2 = g + 1 > g - 1,$$

[1]Each p_j is a ramification point of the hyperelliptic map $\pi : C \to \mathbb{P}^1$. Thus $\pi^{-1}(\pi(p_j)) = 2p_j$ and consequently $2p_j \sim g^1_2$ (Here g^1_2 is the pullback of $\mathcal{O}_{\mathbb{P}^1}(1)$ along π). Therefore, if $D_1 = \sum_{j=1}^{d} p_j$, then $2D_1 = \sum_{j=1}^{d} 2p_j \sim d \cdot g^1_2$. Therefore, $2D_1$ is a multiple of the unique g^1_2.

together with Riemann–Roch this implies that $h^0(C, \mathcal{O}_C(D+E)) > 0$. Now starting from an effective divisor linearly equivalent to $D + E$, we can repeatedly peel off fibers of the hyperelliptic map, i.e. whenever a conjugate pair appears, it is linearly equivalent to E. Extracting as many copies as possible we can thus write

$$D \sim mE + F.$$

Lemma 3.0.5 shows that a divisor F supported on branch points is determined by its linear equivalence class unless $\deg F = g + 1$, in which case it is also linearly equivalent to the complementary sum of branch points by lemma 3.0.4. Hence the expression $D \sim mE+F$ is unique except when $m = -1$. Taking degrees, we get that

$$g - 1 = \deg D = 2m + \deg F.$$

Therefore $\deg F = g - 1 - 2m$ and consequently F is the sum of $g - 1 - 2m$ distinct points p_i. Because $\deg F \geqslant 0$ we thus get that

$$-1 \leqslant m \leqslant (g - 1)/2.$$

So we have found that for each fixed m, the divisor F is determined by a choice of $g - 1 - 2m$ distinct branch points among the $2g + 2$ branch points. By uniqueness, different choices give different theta characteristics, except when $m = -1$. Thus for $m \neq -1$ we get

$$\binom{2g+2}{g-1-2m}$$

theta characteristics and when $m = -1$, we get

$$\frac{1}{2}\binom{2g+2}{g+1}$$

theta characteristics. Therefore, the number of theta characteristics can be written as a sum of binomial coefficients. In particular, if g is odd, then the count is equal to

$$\binom{2g+2}{0} + \binom{2g+2}{2} + \binom{2g+2}{4} + \cdots + \binom{2g+2}{g-1} + \frac{1}{2}\binom{2g+2}{g+1}.$$

If g is even, then it is equal to

$$\binom{2g+2}{1} + \binom{2g+2}{3} + \binom{2g+2}{5} + \cdots + \binom{2g+2}{g-1} + \frac{1}{2}\binom{2g+2}{g+1}.$$

Now note that either row is just the sum of every other entry in the $(2g + 2)$-nd row of Pascal's triangle where we start adding up from the left and ending with half of the middle

term. The sum we get is then just half of the sum of every other entry in the whole row. The first identity in lemma 2.8.1 thus tells us that it is equal to

$$\frac{1}{4}2^{2g+2} = 2^{2g}.$$

Now note that by definition of a theta characteristic $D \sim mE + F$, the parity is equal to the parity of $h^0(\mathcal{O}_C(mE + F))$. Since F cannot vary in the linear equivalence class of $mE + F$, i.e. every effective divisor linearly equivalent to $mE + F$ has to contain F, the only thing that can vary are the points of mE. Since mE can be interpreted as the pullback of a degree m divisor on $\mathbb{P}^1$, we get that $h^0(\mathcal{O}_C(mE)) = m + 1$ and thus that

$$h^0(\mathcal{O}_C(mE + F)) = m + 1.$$

Therefore we can just calculate the number of even and odd theta characteristics respectively by picking alternating terms in the sums above. Then identities (2) and (3) in lemma 2.8.1 tell us that there are

$$2^{g-1}(2^g + 1)$$

even and

$$2^{g-1}(2^g - 1)$$

odd theta characteristics. □

[2]It is unique because for $g \geq 2$ a hyperelliptic curve admits a unique degree-2 map to $\mathbb{P}^1$ and thus a unique degree-2 pencil.

Chapter 4

Bitangents on Nonsingular Plane Quartics

In this chapter we will apply the theory of theta characteristics to an enumerative problem in the geometry of nonsingular quartic plane curves. In particular, we will prove a correspondence between the odd theta characteristics on smooth quartic plane curves and their bitangents and hence show that their number is independent of the particular curve. We begin by defining what precisely we mean by a bitangent to a curve. The main sources for this chapter were [EH24], [Dol12], and [Har77].

Definition 4.0.1 (Bitangent)**.** *Let $C \subset \mathbb{P}^2$ be a smooth plane curve. By a bitangent to C we mean a line $L \subset \mathbb{P}^2$ which is either tangent to C at two distinct points or touches C with order at least 4 at some point.*

While this formulation is probably the best in order to get an intuitive grasp of bitangents, it is useful for our purposes to define bitangents more abstractly in the language of divisors.

Definition 4.0.2. *A bitangent is a line corresponding to an effective divisor $D = p + q$ of degree 2 on C such that $C.L \geqslant 2D$.*

It is not difficult to see that these two definitions are equivalent. In the case that L is tangent to C in two distinct points we just have $D = p + q$ for $p \neq q$ and in the case that L touches C with order at least 4 we have $D = 2p$. Below is an example of the so-called Edge Quartic and all of its bitangents in the real affine chart at $z = 0$.

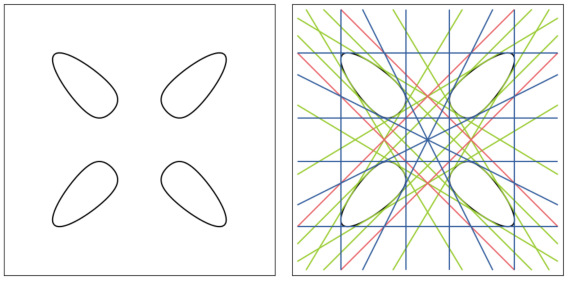


Figure 4.1: For the *Edge Quartic*

$$25(x^4 + y^4 + z^4) - 34(x^2y^2 + x^2z^2 + y^2z^2) = 0$$

all bitangents lie in the real plane of the affine chart at $z = 0$ [BB24]

.

Why in particular do we care about quartic curves? We can quite easily see that smooth conic and cubic curves cannot have any bitangents at all because Bézout's theorem 2.1.2 tells us that a line intersects a conic in exactly two points (with multiplicity) and a conic in exactly three points (with multiplicity). Bitangents only exist and thus become interesting for curves of degree $\geqslant 4$. If C is quartic, then $C.L$ has degree 4. Thus the inequality in the definition above forces $C.L = 2p + 2q$ for a bitangent. For the remainder of this chapter we will talk about quartic smooth plane curves. First let us state the main result.

Proposition 4.0.3 (Bitangent Theta Characteristics Correspondence)**.** *If $C \subset \mathbb{P}^2$ is a smooth quartic plane curve, then the map*

$$\beta : L \mapsto \frac{1}{2}(C \cdot L)$$

gives a bijection of sets

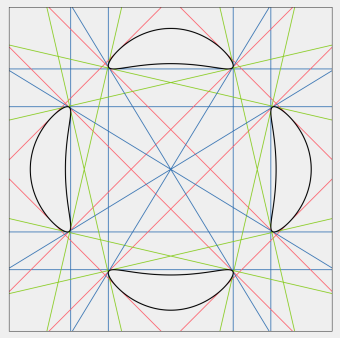
$$\{\textit{bitangents on } C\} \leftrightarrow \{\textit{odd theta characteristics on } C\}.$$

We will first discuss the consequences of this proposition before we proceed to prove it at the end of the chapter.

This proposition allows us to reduce the bitangent count to the count of odd theta characteristics, a problem that we already solved in the previous chapter. Note that a smooth quartic plane curve has genus 3 by lemma 2.1.3. By theorem 3.0.2 a curve of genus g has $2^{g-1}(2^g - 1)$ odd theta characteristics. Setting $g = 3$ gives $4 \cdot 7 = 28$. Thus proposition 4.0.3 implies the following result.

Corollary 4.0.4. *If $C \subset \mathbb{P}^2$ is a smooth quartic plane curve, then it has exactly 28 bitangents.*

Sit back for a second to appreciate what this result tells us: the bitangent count of a smooth quartic curve is completely independent of the curve itself. This seems quite counterintuitive, given that bitangents are primarily geometric objects. For example, compare the *Blum-Guinand quartic* and its bitangents below to the *Edge Quartic* we already saw earlier.



$$\frac{50625}{638}\left[\left(\frac{638+2\sqrt{9889}}{450}x^2+\frac{638-2\sqrt{9889}}{450}y^2-1\right)\left(\frac{638-2\sqrt{9889}}{450}x^2+\frac{638+2\sqrt{9889}}{450}y^2-1\right)+\frac{13}{625}\right]=0$$

Figure 4.2: For the *Blum-Guinand quartic* all bitangents lie in the real plane of the affine chart at $z = 0$ [BG64][1]

.

Now let us prove the correspondence that allows us to count the number of bitangents to a curve by counting the number of odd theta-characteristics.

First we want to show the following lemma.

Lemma 4.0.5. *Let C be a smooth quartic plane curve, then the bitangents are in one-to-one correspondence with the effective divisors D with $2D \sim K_C$.*

Proof. For a smooth quartic plane curve C Bezout's theorem tells us that the intersection of C with a line L consists of 4 points (with multiplicity). Thus the inequality $C.L \geqslant 2D$ becomes an equality and we get $C.L = 2p + 2q$. Thus the divisor $2D$ is the complete intersection of C and L. Since C is a smooth plane quartic, adjunction 2.3.13 [2] gives

$$K_C \sim (K_{\mathbf{P}^2} + C)|_C = (-3H + 4H)|_C = H|_C.$$

where H denotes the class of a line in $\mathbb{P}^2$, so $H\ |_C$ is the divisor cut out on C by a line. Therefore,

$$2D \sim H|_C \sim K_C.$$

[1] An equivalent and slightly cleaner form of this curve is $144(x^4+y^4)+350x^2y^2-225(x^2+y^2)+81 = 0$ but we use the present form because it is closer to that of Blum and Guinand.

Conversely, let D be an effective divisor such that $2D \sim K_C$. Then $K_C \sim H_C$ implies that

$$2D \sim H_C$$

which means that there is a line L such that $C.L = 2D$. Thus L is a bitangent to C.

□

Lemma 4.0.6. *The effective divisors of C with $2D \sim K_C$ are exactly the ones corresponding to the odd theta characteristics of C.*

Proof. The genus g of C is 3 by lemma 2.1.3. Thus the degree formula 2.3.9 for K_C tells us that every theta characteristics $\mathcal{L}$ on C has degree

$$\deg \mathcal{L} = \frac{\deg K_C}{2} = \frac{2g-2}{2} = 2.$$

If D is an effective divisor with $2D \sim K_C$, then $\mathcal{L} = \mathcal{O}_C(D)$ is a theta characteristic and because D is effective, we have $\operatorname{div}(1) + D = D \geq 0$. So 1 defines a global section of $\mathcal{O}_C(D)$. Thus $H^0(C, \mathcal{O}_C(D)) \neq 0$, and consequently $h^0(\mathcal{L}) \geq 1$. But we cannot have $h^0(\mathcal{L}) \geqslant 2$ because then $\mathcal{L}$ would define a g^1_2 which implies that C is hyperelliptic which is a contradiction to lemma 2.1.6. Therefore, $h^0(\mathcal{L}) = 1$ and we get that effective divisors of the assumed form correspond to odd theta characteristics.

Conversely, assume $\mathcal{L}$ to be an odd theta characteristic. The same argument as above tells us that $h^0(\mathcal{L}) \leqslant 1$ and therefore $h^0(\mathcal{L}) = 1$. But this implies that

$$\mathcal{L} \cong \mathcal{O}_C(D)$$

for some effective divisor D with degree 2. Since $\mathcal{L}^{\otimes 2} \cong \omega_C$, we then get that

$$2D \sim K_C.$$

□

With these two lemmas in hand, the proof of proposition 4.0.3 falls right out.

Proof. By lemma 4.0.5 the bitangents of C are in one-to-one correspondence with the effective divisors of C that satisfy $2D \sim K_C$ and by 4.0.6 these effective divisors are in one-to-one correspondence with the odd theta characteristics of C. By combining these two correspondences, we thus get a one-to-one correspondence between the bitangents of C and its odd theta characteristics. □

[2]Here we use that $K_{\mathbb{P}^2} = \mathcal{O}_{\mathbb{P}^2}(-3) \cong \mathcal{O}_{\mathbb{P}^2}(-1)^{\otimes 3}$ where $\mathcal{O}_{\mathbb{P}^2}(-1)$ is the dual sheaf of the hyperplane line bundle $\mathcal{O}_{\mathbb{P}^2}(1)$ whose sections are exactly the linear forms in the homogeneous coordinates x_0, x_1, x_2. For a more in-depth discussion see chapter 5 in [Har77]. Note also that C being a quartic implies that it is 4 times the line class.

Chapter 5

Counting Theta Characteristics on Singular Curves

In this chapter we will discuss the generalization of theta characteristics to singular curves which was initially developed by [Har82]. We found earlier that a smooth curve of genus g has 2^{2g} theta characteristics of which $2^{g-1}(2^g+1)$ are even and $2^{g-1}(2^g-1)$ are odd. It turns out that in the case of singular curves these counts are entirely determined by the singularities of the curve. The only restriction on the curves we consider is that they are Gorenstein.

Definition 5.0.1 (Gorenstein Curve)**.** *We call a curve C Gorenstein if its dualizing sheaf is invertible, i.e. a line bundle.*

An equivalent definition for a Gorenstein curve C is that the local rings $\mathcal{O}_{C,p}$ of C are all Gorenstein rings.

There are many examples of Gorenstein curves. For instance, any local complete intersection curve 2.1.8 and thus any curve on a smooth surface is a Gorenstein curve.

Example 5.0.2 (Examples of Singular Gorenstein Curves)**.** *All of the classic examples of singular curves like node, cusp, and tacnode are Gorenstein.*

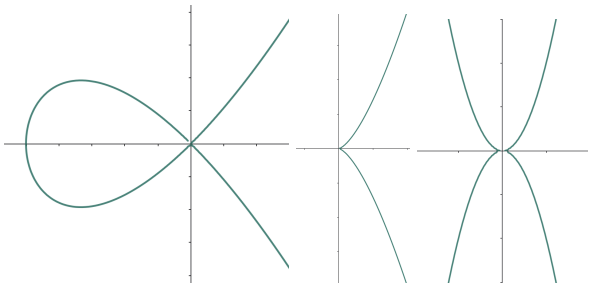


Figure 5.1: Affine charts at $z = 1$ of the node $y^2z = x^3 + x^2z$, the cusp $y^2z = x^3$, and the tacnode $y^2z^2 = x^4$ [Har77]

.

While the conditions on Gorenstein curves are not extremely restrictive, they do exclude certain tame curves such as the union of the three coordinate axes in $\mathbb{A}^3$.

We will now present the main result of this chapter. The reader might be unfamiliar with some of the objects in the theorem statement. These will be provided below, when we dissect the result. In its essence, it describes how singularities determine the number of theta characteristics on a curve. In the statement both of the variables l and k are integers dependent on the curve C. We will define them properly right after stating the theorem.

Theorem 5.0.3. *Let C be a Gorenstein curve, $\pi : \widetilde{C} \to C$ its normalization 7.1.10 and $\widetilde{g}$ the sum of the genera of the components of $\widetilde{C}$.* [1]

1. *If $l \not\equiv 0$, then there are $2^{2\widetilde{g}+k-1}$ theta characteristics of which exactly half are even and half are odd.*

2. *If $l \equiv 0$, then there are $2^{2\widetilde{g}+k}$ theta characteristics and they split either into*

$$2^{\widetilde{g}+k-1}(2^{\widetilde{g}}+1) \text{ even}, \qquad 2^{\widetilde{g}+k-1}(2^{\widetilde{g}}-1) \text{ odd}$$

or into

$$2^{\widetilde{g}+k-1}(2^{\widetilde{g}}-1) \text{ even}, \qquad 2^{\widetilde{g}+k-1}(2^{\widetilde{g}}+1) \text{ odd}$$

theta characteristics.

There are two variables in this theorem that require further attention: l and k .

We will start with l. To define it, we will first need to introduce the concept of a theta form. To begin, recall that J_2 is the space of torsion 2 divisor classes 2.6.1.

Definition 5.0.4 (Theta-Form)**.** *Let $\mathcal{L}$ be a theta characteristic on the curve C. Then we call the map q_L defined by*

$$q_L : J_2 \to \mathbb{Z}/2, \qquad [D] \mapsto h^0(\mathcal{L}(D)) - h^0(\mathcal{L}) \quad \text{modulo } 2$$

[1] For a discussion on how to compute these genera see section 53.8 in [Sta26] and [Har82].

a theta-form.

In other words, a theta-form is a map that sends each 2-torsion divisor class $[D] \in J_2$ to the parity of the change in the dimension of global sections when the theta-characteristic $\mathcal{L}$ is twisted by $\mathcal{O}_C(D)$, where $\mathcal{O}_C(D)$ is the line bundle whose sections are meromorphic functions f on C satisfying $\operatorname{div}(f) + D \geq 0$.

It turns out that theta-forms are not just any functions; indeed, they are quadratic forms modulo 2 with a well-known bilinear form associated to them. Recall that a quadratic form is a symmetric bilinear form $V \times V \to \mathbb{C}$ on a $2n$-dimensional vector space V.

Theorem 5.0.5 (Riemann-Mumford Relation)**.** *The function q_L is a modulo 2 quadratic form on J_2 and its associated bilinear form is the Weil pairing λ which we defined in 2.6.2.*

Next, we will want to understand what will turn out to be the nullspace of this quadratic form $q_{\mathcal{L}}$. We will assume from now on that $\pi : \widetilde{C} \to C$ is the normalization of the curve C. Then we have an exact sequence of sheaves given by

$$0 \to \mathcal{O}_C^* \xhookrightarrow{\iota} \pi_* \mathcal{O}_{\widetilde{C}}^* \to \mathcal{F} \to 0 \tag{5.0.1}$$

where $\mathcal{O}_C^*$ is the sheaf of nowhere vanishing regular functions on C, and $\pi_* \mathcal{O}_{\widetilde{C}}^*$ is the pushforward along π of the nowhere vanishing regular functions on $\widetilde{C}$, i.e. $\pi_* \mathcal{O}_{\widetilde{C}}^*(U) = \mathcal{O}_{\widetilde{C}}^*(\pi^{-1}(U))$ for each open set $U \subseteq C$. The map ι is defined by

$$\mathcal{O}_C^*(U) \ni \quad f \overset{\iota}{\mapsto} f \circ \pi \quad \in \mathcal{O}_{\widetilde{C}}^*(\pi^{-1}(U)) = \pi_* \mathcal{O}_{\widetilde{C}}^*(U).$$

To check injectivity, suppose that $f \circ \pi = g \circ \pi$. Since π is surjective, this implies that $f = g$. The sheaf $\mathcal{F}$ in this short exact sequence is, colloquially speaking, the sheaf of local gluing defects for invertible functions under normalization. More precisely, for any singular point $p \in C$, the stalk of $\mathcal{F}$ at p is

$$\mathcal{F}_p = \left(\pi_* \mathcal{O}_{\widetilde{C}}^*\right)_p / \mathcal{O}_{C,p}^*$$

where $\left(\pi_* \mathcal{O}_{\widetilde{C}}^*\right)_p$ consists of independent elements for each of the branches $p_1, \ldots, p_r \in \pi^{-1}(p) \subset \widetilde{C}$. Thus $\mathcal{F}_p$ measures the failure of the branchwise elements to come from an element of the singular local ring. On any nonsingular point π is an isomorphism between $\left(\pi_* \mathcal{O}_{\widetilde{C}}^*\right)_p$ and $\mathcal{O}_{C,p}^*$ and thus $\mathcal{F}$ is 0 on these points. In other words, $\mathcal{F}$ is supported precisely at the singular points of C.

Now we can apply sheaf cohomology to the short exact sequence of sheaves to get a long exact sequence of sheaf cohomology groups over C.

$$0 \to H^0\left(\mathcal{O}_C^*\right) \to H^0\left(\pi_*\mathcal{O}_{\widetilde{O}}^*\right) \to H^0(\mathcal{F}) \to H^1(\mathcal{O}_C^*) \to H^0\left(\pi_*\mathcal{O}_{\widetilde{O}}^*\right) \to H^0(\mathcal{F}) \to \dots$$

Since $\mathcal{F}$ is only supported at finitely many points, we get that $H^1(\mathcal{F}) = 0$ by lemma 7.1.11.

Now by lemma 2.7.1 we can shorten our exact sequence to get the short exact sequence

$$0 \to H^0(\mathcal{F})/\operatorname{Im} H^0\left(\pi_*\mathcal{O}_{\widetilde{C}}^*\right) \to H^1(\mathcal{O}_C^*) \to H^1\left(\pi_*\mathcal{O}_{\widetilde{C}}^*\right) \to 0$$

Now note that [2]

$$H^1(\mathcal{O}_C^*) = \text{Pic}(C) \qquad \text{and} \qquad H^1\left(\pi_*\mathcal{O}_{\widetilde{C}}^*\right) = H^1\left(\mathcal{O}_{\widetilde{C}}^*\right) = \text{Pic}\left(\widetilde{C}\right)$$

Thus after restricting to degree 0 and denoting the quotient $H^0(\mathcal{F})/\operatorname{Im} H^0\left(\pi_*\mathcal{O}_{\widetilde{C}}^*\right)$ by Γ, we get the short exact sequence

$$0 \to \Gamma \to J(C) \xrightarrow{\pi^*} J(\widetilde{C}) \to 0.$$

Note that we can interpret Γ as the group of line bundles $\mathcal{L}$ whose pullback $\pi^*\mathcal{L}$ is trivial. Finally, the short exact sequence above also induces an exact sequence on the subgroups of elements of order 2 in the following way. Consider the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Gamma & \longrightarrow & J(C) & \longrightarrow & J(\widetilde{C}) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \times 2} & & \downarrow{\scriptstyle \times 2} & & \downarrow{\scriptstyle \times 2} & & \\ 0 & \longrightarrow & \Gamma & \longrightarrow & J(C) & \longrightarrow & J(\widetilde{C}) & \longrightarrow & 0 \end{array}$$

As it is clearly commutative, the snake lemma 2.7.2 gives us the exact sequence

$$0 \to \ker\left(\Gamma \xrightarrow{\times 2} \Gamma\right) \to \ker\left(J(C) \xrightarrow{\times 2} J(C)\right) \to \ker\left(J(\widetilde{C}) \xrightarrow{\times 2} J(\widetilde{C})\right)$$

which we can rewrite as

$$0 \to \Gamma_2 \to J_2(C) \to J_2(\widetilde{C}),$$

the exact sequence on the subgroups of 2-torsion elements. Now the definition of the Weil pairing 2.6.2 immediately implies the following result.

Proposition 5.0.6. *Γ_2 is the nullspace of the bilinear form λ.*

[2] We will not prove this here since we have decided to defer the study of cohomology to the reader for this thesis. The interested reader should refer to lemma 6.1 and section 20.2 in [Sta26].

Similarly elementary is the following fact.

Proposition 5.0.7. *The restriction $q_{\mathcal{L}} \mid_{\Gamma_2}$ of $q_{\mathcal{L}}$ to $\Gamma_2 \subset J_2(C)$is a linear functional and it is independent of $\mathcal{L}$.*

Now we are finally able to define what we meant by l in the statement of theorem 5.0.3.

Definition 5.0.8. *We call $q_{\mathcal{L}} \mid_{\Gamma_2}$ the linear part of the theta form $q_{\mathcal{L}}$ and denote it by l.*

The other undefined variable in the problem statement was k which we can now describe, having introduced Γ_2. It is defined as the rank of Γ_2 as a vector space over $\mathbb{Z}/2$, i.e. $k := \dim_{F_2} \Gamma_2$.

Now we have seen what l, the case deciding variable of the theorem, is defined as. But how does it depend on the singularities of the curve C? The following theorem gives us the answer.

Theorem 5.0.9. *Let C be a Gorenstein curve, $l : \Gamma_2 \to \mathbb{Z}/2$ the linear part of the theta-form on C, $\pi : \widetilde{C} \to C$ the normalization of C, and D the adjoint divisor on $\widetilde{C}$. If $p \in C$ is any singularity, $q \in \pi^{-1}(p)$, and e_q is the corresponding element of Γ_2, then*

$$l(e_q) = \textit{mult}_q(D).$$

We have previously already seen all objects in this statement except for one: e_q. So let us construct and define it.

As in the problem statement, let $p \in C$ be a singular point and $\pi : \widetilde{C} \to C$ be the normalization of C. Let $\pi^{-1}(p) = \{q_1, \ldots, q_b\}$ be the b branch points over p. Recall that for $\mathcal{F}$ from the short exact sequence 5.0.1 the group of sections $\mathrm{Sec}(\mathcal{F}_p)$ measures how a line bundle can be obtained by gluing the branches over p with multiplicative constants.

Definition 5.0.10. *Write Γ_p to denote the stalk defined by the torsion 2 subgroup of* $\mathrm{Sec}(\mathcal{F}_p)$.

Since each element of $\mathrm{Sec}(\mathcal{F}_p)$ can be represented by an invertible function f on the normalization near the branch points q_i, we can construct a map

$$\psi : \mathrm{Sec}(\mathcal{F}_p) \to (\mathbb{C}^*)^{b-1}, \qquad f \mapsto \left(\frac{f(q_1)}{f(q_b)}, \frac{f(q_2)}{f(q_b)}, \ldots, \frac{f(q_{b-1})}{f(q_b)} \right).$$

The kernel of ψ consists of classes represented by functions that have the same value on every branch. Up to a scalar that value is equal to 1 on every branch. Thus it lies in the principal unit group $1 + \mathfrak{m}_p$. But that group has no nontrivial torsion in characteristic 0 as can be easily seen by exponentiating its nontrivial elements. This implies that the kernel

of ψ is entirely torsion free. Basic group theory and the fact that every torsion 2 element of $(\mathbb{C}^*)^{b-1}$ actually lifts[3] to a torsion 2 element of $\mathrm{Sec}(\mathcal{F}_p)$ imply that we have a one-to-one identification between these two torsion 2 subgroups. But the order 2 elements of $(\mathbb{C}^*)^{b-1}$ are famously exactly the elements of the form $(\pm 1, \pm 1, \ldots, \pm 1)$ which implies that

$$\Gamma_p \cong (\mathbb{Z}/2)^{b-1}.$$

Now for each branch q_i we define the class e_{q_i} in the following way.

Definition 5.0.11. *We use e_{q_i} to denote the class of the image in Γ_2 of the sections of $\pi_* \mathcal{O}^*_{\widetilde{C}}$ which assume the value -1 in neighborhood of q_i and 1 in a neighborhood of q_j with $j \neq i$.*

Equivalently, e_{q_i} is the line bundle constructed from the trivial bundle on $\widetilde{C}$ by identifying the fibers at the points q_α via multiplication by -1 on L_{q_i}. Then Γ_p is generated by these e_{q_i} modulo the relation

$$\sum_{q \in \pi^{-1}(p)} e_q = 0.$$

Now that we understand the theory, let us apply it to some simple examples of singular curves.

Corollary 5.0.12. *If the curve C has a node, then the theta characteristics split equally into even and odd, i.e.*

$$\#S^+ = \#S^- = \frac{1}{2}\#S.$$

Proof. Let p be the ordinary node of C and q a preimage point of p. Then theorem 5.0.9 tells us that $l(e_q) = 1$. This means that $l \not\equiv 0$ and thus theorem 5.0.3 implies that the theta characteristics split evenly. □

[3]Take a local unit on $\widetilde{C}$ equal to 1 on some branches and -1 on the others.

Chapter 6

The Apollonius Problem

Theta characteristics on smooth curves allowed us to solve the geometric bitangent count problem for quartic plane curves. The extension of the theory of theta characteristics to singular curves allows us to solve an array of additional problems. In this section we will look at one of them, namely the Apollonius problem. To the best of our knowledge this application was first done in [Har82]. Classically, the problem is stated as follows.

Classical Apollonius Problem: *If C_1, C_2, C_3 are three circles in the plane, what are the circles that are tangent to all three of the circles C_i?*

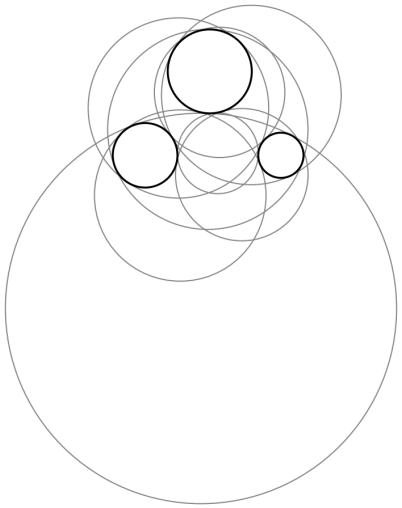


Figure 6.1: Apollonius Configuration. The three original circles are in black, the tangent circles are in grey.

Interestingly, this can be translated into the language of theta characteristics and solved there by using a similar odd theta characteristics correspondence as we saw when counting bitangents of quartic plane curves. In particular, we will construct a curve C out of the three circles C_1, C_2, C_3 and show that for this curve there are one-to-one correspondences

$$\begin{aligned}\{\text{odd theta characteristics on } C\} &\leftrightarrow \{\text{effective semicanonical divisors on } C\} \\ &\leftrightarrow \{\text{circles tangent to } C_1, C_2, C_3\}.\end{aligned}$$

Theorem 6.0.1. *Let C_1, C_2, C_3 be circles, and E_1, E_2, E_3 be circles tangent to each of the C_i. Then there is a bicircular[1] quartic curve F that passes through all 9 contact points $E_i \cdot C_j$ such that the residual points of the intersection of F with the C_i are the contact points of a fourth circle E_4 that is tangent to each of the C_i.*

The proof of this theorem will require a few additional definitions and facts about canonical divisor that we will state without proof. The interested reader can find proofs in the associated sources.

Definition 6.0.2 (Adjoint Curve)**.** *Let C and A be planar curves. We call A and adjoint curve to C if*

$$\mathit{mult}_p(A) \geqslant \mathit{mult}_p(C) - 1$$

for every (multiple) point $p \in C$ [Ful08].

Let us spell out for a moment what this definition really means. If $p \in C$ is a singular point, then $\text{mult}_p(C)$ is at least 2 which translates into

$$\text{mult}_p(A) \geqslant 1,$$

i.e. that p lies on A. For any nonsingular point of C we have $\text{mult}_p(C) = 1$ and the inequality becomes

$$\text{mult}_p(A) \geqslant 0$$

which is vacuoulsy true for any point of the projective plane. That means the definition really only says that an adjoint curve A has to pass through all singular points of C. The following fact about adjoint curves will come in handy during the proof of the theorem.

Fact 6.0.3. *Let C be a curve of degree d. Then the adjoint curves of degree $d-3$ cut out the complete canonical series [Gor52].*

Proof. Consider $\mathbb{P}^2$ with coordinates X_0, X_1, X_2. Take an affine chart with $X_0 = 0$ as the line at infinity. Then circles in this charts are conic curves passing through the line

[1] By *bicircular we mean that it passes through both circular points $[0:1:i]$ and $[0:1:-i]$ through which any affine circle has to pass in its projective representation. For a further discussion of this see the first paragraph of the proof.*

at infinity at the two *circular* points $[0:1:i]$ and $[0:1:-i]$. To see this note that a general affine circle is solution set of $x^2+y^2+ax+by+c=0$ for some $a,b,c\in\mathbb{C}$. Homogenizing this with $x=\frac{X_1}{X_0}$ and $y=\frac{X_2}{X_0}$ gives $X_1^2+X_2^2+aX_0X_1+bX_0X_2+xX_0=0$ which reduces to $0=X_1^2+X_2^2=(X_1+iX_2)(X_1-iX_2)$ for $X_0=0$.

For each of the three given circles C_i let f_i be the polynomial cutting it out. Then the sextic curve $C'=Z(f_1\cdot f_2\cdot f_3)$ is the union of the three circles C_i and] the circular points both have multiplicity 3 on C' as each of the C_i intersects them. Two generic circles intersect each other in two points. Thus C' generically has multiplicity 2 at 6 additional points in the affine chart.

Now let C be the curve constructed from C' by normalizing the two circular points at infinity. By lemma **??** C is a curve of genus 4. Since C has degree 6, the useful fact 6.0.3 tells us that the adjoint curves of degree 3 cut out the complete canonical series of C. Because the circular points both have multiplicity 3, we then get that an adjoint curve has to have mutliplicity at least 2 at each of them. In summary, the adjoint curves are cubics that are double at the two circular points.

Now we claim that that there is a one-to-one correspondence between such cubics and affine circles. To show this, we first prove that each such cubic X must contain the line at infinity L_∞ as a component; if L_∞ were not a component, Bezout's theorem 2.1.2 tells us that

$$\deg X\cdot\deg L_\infty=3.$$

But X intersects L_∞ with multiplicity 2 at either circular point and since $2+2=4\neq 3$ we get a contradiction. Thus L_∞ divides X with a residual conic Y that intersects L_∞ exactly at the two circular points and by the argument we saw in the first paragraph of this proof we get that such Y are exactly the projective characterization of a circle.

By a similar argument to the complete canonical series of C being cut out by the adjoint curves of degree 3 we see that the bicanonical series $|2K_C|$ is cut out on C by quartics that have multiplicity 2 on each of the two circular points.

Let Q be the surface we get from blowing up the two circular points and blowing down the line at infinity to a point p_∞. Then we can embed Q as a smooth quadric in $\mathbb{P}^3$ using the Segre embedding with image $X_0X_3-X_1X_2=0$ [2]. Given this embedding, consider a plane intersection $H\cap Q$ for some plane H and project each point q of this intersection to $\mathbb{P}^2$ through the ray $\overline{p_\infty q}$. The image of any such projection will be a circle because the projection maps a conic on Q to a conic in the plane that goes through the circular points which forces the conic to be a circle. Then C is just the transform of $C_1\cup C_2\cup C_3$ in Q, i.e. the union of three plane sections of Q. In other words, if we write H_i for the plane whose intersection with Q is the transform of C_i, then we can write

$$C=Q\cap(H_1\cup H_2\cup H_3).$$

So it is a complete intersection of a quadric and a cubic in $\mathbf{P}^3$. But by Petri's theorem 2.1.4curves of type $(2,3)$ in $\mathbb{P}^3$ are exactly the canonical genus 4 curves. Thus we represented C a canonical curve of genus 4.

C is embedded in $\mathbb{P}^2$ and thus nonhyperelliptic. For a nonhyperelliptic curve of genus 4, every g^1_3 arises from one of the two rulings [3] of Q. In either of the two rulings a line meets C in a divisor of degree 3 by Bezout's theorem 2.1.2. These divisors vary in a pencil and under the projections from p_∞, the two rulings correspond precisely to the two pencils of lines passing through the two circular points. Therefore, there are no other pencils of degree 3 on C then the ones cut out by these two line-pencils.

Now assume D to be a semicanonical divisor on C, i.e. $2D \sim K_C$. By lemma 2.2.13 we have $\deg K_C = 2g-2 = 6$, and thus D must have degree $\frac{\deg K_C}{2} = 3$. Therefore, a semicanonical divisor moving in a pencil, i.e. a divisor that is not isolated in its linear series, would define a g^1_3. This contradicts what we found in the previous paragraphs as the only two degree 3 pencils are the ones described in the previous paragraph and semicanonical divisors cannot vary in either of them as there are only finitely many theta characteristics. This means that every effective semicanonical divisor is isolated. By Clifford's theorem, the line bundle $\mathcal{L}$ of a semicanonical divisor of degree 3 satisifies $h^0(\mathcal{L}) \leqslant 2$. Thus, if it does not move in a pencil, then $h^0(\mathcal{L}) = 1$ which means that the theta characteristic $\mathcal{L}$ is odd. Therefore, we have a one-to-one correspondence between odd theta-characteristics on C and the effective semicanonical divisors.

Since we found that the canonical series is cut out by circles, any semicanonical divisor can be represented by a circle whose intersections with C all have even multiplicity. By definition of C, this means that in the generic Apollonius situation the circle is tangent to each of the three C_i. Thus, the effective semicanonical divisors correspond exactly to the solutions of Apollonius problem.

Now we have

$$\Gamma_2 = J_2 \cong H_1(C, \mathbb{Z}/2) \cong (\mathbb{Z}/2)^4.$$

Theorem 5.0.9 tells us that $l \not\equiv 0$ and thus we can conclude that there are exactly eight solutions to Apollonius problem since we found that the tangent circles correspond to effective canonical divisors of C. Finally, corollary **??** implies that these solutions form a coset of the subgroup of J_2 in S. Consequently, any triple of effective semicanonical divisors on C comes with a fourth one such that their sum equals $2K_C$ which is the statement of the theorem.

□

[2] Up to isomorphism Q is $\mathbb{P}^1 \times \mathbb{P}^1$ and that is its image under the Segre map.

[3] To see that there are only two rulings of Q note that the tangent plane T_pQ meets Q in a singular conic. Indeed, because T_pQ is tangent to Q at p, the plane section has multiplicity at least 2 at p. Thus p is a singular point of that conic. Since Q is smooth, this conic cannot be a double line. Therefore, it must be the union of two distinct lines through p. Thus, through each point of Q we have exactly two lines contained in

Although the Apollonius problem can be solved by classical methods, the theta-characteristic approach gives a deeper understanding of the configuration and allows us to prove related results such as Hart's theorem as discussed in [Har82].

Q. Since a ruling on Q is a 1-parameter family of lines on Q such that we have exactly one line from that family through each point, at any point we have at most two choices. Therefore Q can have at most two rulings.

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Chapter 7

Appendix

7.1 Schemes

We introduce schemes in order to define what we mean by the normalization of a curve. In the process we also introduce the notion of a spectrum which is useful for the definition of the structure sheaf 2.2.14. At the end we also state a lemma about coherent sheaves that we used in chapter 5. Most of this section is from [Vak25]. We begin by defining the spectrum of a ring.

Definition 7.1.1 (Spectrum of a Ring)**.** *Let A be a ring. As a set the spectrum of A, denoted by $\operatorname{Spec} A$ is the set of all prime ideals of A.*

For clarity of notation, we write $[\mathfrak{p}]$ when we interpret a prime ideal $\mathfrak{p}$ as an element of $\operatorname{Spec} A$. We call elements of A functions on $\operatorname{Spec} A$ and we we evaluate $f \in A$ at $[\mathfrak{p}]$ by taking $f \pmod{\mathfrak{p}}$.

The topology on $\operatorname{Spec} A$ is the Zariski topology generated by the base of distinguished sets; for $f \in A$ the distinguished open set of D is defined as

$$D(f) := \{[\mathfrak{p}] \in \operatorname{Spec} A : f \notin \mathfrak{p}\} = \{[\mathfrak{p}] \in \operatorname{Spec} A : f([\mathfrak{p}]) \neq 0\},$$

i.e. the locus where f does not vanish.

To define a scheme we also have to introduce ringed spaces.

Definition 7.1.2 (Ringed Space)**.** *Let $\mathcal{O}_X$ be a sheaf of rings on a topological space X. Then we call the tuple $(X, \mathcal{O}_X)$ a ringed space.*

Definition 7.1.3 (Isomorphism of Ringed Spaces)**.** *By an isomorphism of ringed spaces we mean the following data.*

1. *A homeomorphism $\pi : X \to Y$ of the topological spaces, together with*

2. *an isomorphism of sheaves $\mathcal{O}_Y \xrightarrow{\cong} \pi_* \mathcal{O}_X$.*

Intuitively, an isomorphism of ringed spaces provides a one-to-one correspondence of the sets, topologies and structure sheaves of two ringed spaces.

Now we are ready to define affine schemes which in turn will allow us to define the general notion of a scheme.

Definition 7.1.4 (Affine Scheme)**.** *We call a ringed space $(X, \mathcal{O}_X)$ an affine scheme if it is isomorphic to $(\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A})$ for some ring A.*

Definition 7.1.5 (Scheme)**.** *We call a ringed space $(X, \mathcal{O}_X)$ a scheme if it has the property that for all points $p \in X$ there exists an open neighborhood $U \ni p$ such that $(U, \mathcal{O}_X \mid_U)$ is an affine scheme.*

Example 7.1.6 (Scheme)**.** *Consider any of the curves discussed in this thesis.*

Now we define some useful vocabulary before defining the normalization of a scheme.

Definition 7.1.7 (Reduced Scheme)**.** *We call a scheme reduced if all of its stalks $\mathcal{O}_{X,p}$ have no nonzero nilpotent elements [Sta26].*

Definition 7.1.8 (Normal Scheme)**.** *We call a scheme normal if all of its local rings are integrally closed domains.*

Definition 7.1.9 (Integral Scheme)**.** *We call a scheme X integral if it is nonempty and if for all nonempty affine open $U \subset X$ the ring R is an integral domain where Spec$(R) = U$.*

Definition 7.1.10 (Normalization of a Scheme)**.** *Let X be any integral scheme and for each open affine subset $U = \operatorname{Spec} A$ of X, denote by $\widetilde{A}$ the integral closure of A in its quotient field. Write $\widetilde{U} = \operatorname{Spec} \widetilde{A}$.*

Then we define the normalization of X to be the normal integral scheme $\widetilde{X}$ constructed by gluing the schemes $\widetilde{U}$ together.

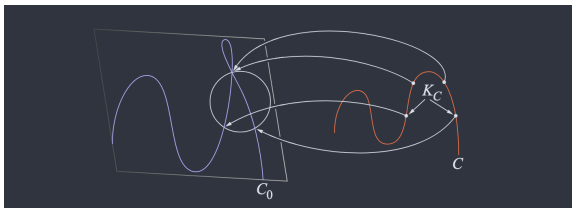

Figure 7.1: Here C is the normalization of a quartic plane curve C_0 with a node and a canonical divisor represented by the conic containing the node [EH24].

Finally, we used the following result in chapter 5.

Lemma 7.1.11. *If X is a locally Noetherian scheme and F a coherent sheaf such that $\dim(\mathrm{Supp}(\mathcal{F})) \leqslant 0$, then $H^i(X, \mathcal{F}) = 0$ for $i \geqslant 0$.*

Proof. For a proof see lemma 30.9.10 in [Sta26]. □

7.2 Combinatorial Identities

Here is the proof for lemma 2.8.1. A shorter sketch of the argument can also be found in [EH24] on page 127.

Proof. 1. This is an immediate consequence of the binomial theorem. Note that

$$2^{2n} = (1+1)^{2n} = \sum_{l=0}^{2n} \binom{2n}{l} \qquad \text{and} \qquad 0 = (1-1)^{2n} = \sum_{l=0}^{2n} (-1)^l \binom{2n}{l}.$$

Thus

$$\begin{aligned} 2^{2n} + 0 &= \sum_{l=0}^{2n} \binom{2n}{l} + (-1)^l \binom{2n}{l} \\ &= \sum_{k=0}^{n} 2\binom{2n}{2k} \end{aligned}$$

and therefore

$$2^{2n-1} = \sum_{k=0}^{n} \binom{2n}{2k}.$$

Similarly,

$$\begin{aligned} 2^{2n} - 0 &= \sum_{l=0}^{2n} \binom{2n}{l} - (-1)^l \binom{2n}{l} \\ &= \sum_{k=0}^{n-1} 2 \binom{2n}{2k+1} \end{aligned}$$

which is equivalent to

$$2^{2n-1} = \sum_{k=0}^{n-1} \binom{2n}{2k+1}.$$

2. First note that

$$(1+i)^{4n} = \left((1+i)^2\right)^{2n} = (2i)^{2n} = (-1)^n 2^{2n}. \tag{7.2.1}$$

But by the binomial theorem we also have that

$$(1+i)^{4n} = \sum_{l=0}^{4n} \binom{4n}{l} i^l.$$

Because the real parts in equation 7.2.1 have to agree, we then get

$$\sum_{k=0}^{n} \binom{4n}{4k} - \sum_{k=0}^{n-1} \binom{4n}{4k+2} = (-1)^n 2^{2n}. \tag{7.2.2}$$

Finally, note that (1) implies that

$$\sum_{k=0}^{n} \binom{4n}{4k} + \sum_{k=0}^{n-1} \binom{4n}{4k+2} = \sum_{k=0}^{4n} \binom{4n}{2k} = 2^{4n-1}. \tag{7.2.3}$$

The identities in (2) now follow immediately from adding and subtracting equation 7.2.2 and 7.2.3.

3. Note that

$$(1+i)^{4n+2} = (2i)^{2n+1} = (-1)^n 2^{2n+1} i \tag{7.2.4}$$

By the binomial theorem the left hand side is equal to

$$\sum_{l=0}^{4n+2} \binom{4n+2}{l} i^l$$

Its imaginary part is equal to

$$\sum_{k=0}^{n} \binom{4n+2}{4k+1} i - \sum_{k=0}^{n-1} \binom{4n+2}{4k+3} i$$

and thus looking at the imaginary parts of equation 7.2.4 gives us the identity

$$\sum_{k=0}^{n} \binom{4n+2}{4k+1} - \sum_{k=0}^{n-1} \binom{4n+2}{4k+3} = (-1)^n 2^{2n+1} \tag{7.2.5}$$

On the other hand, identity (1) implies that

$$\sum_{k=0}^{n} \binom{4n+2}{4k+1} + \sum_{k=0}^{n-1} \binom{4n+2}{4k+3} = \sum_{j=0}^{2n} \binom{4n+2}{2j+1} = 2^{4n+1} \tag{7.2.6}$$

Adding and subtracting equations 7.2.5 and 7.2.6 then returns the wanted identity.

□