

EXISTENCE OF NON-CONFLICTING NOWHERE-ZERO $\mathbb{Z}_2 \times \mathbb{Z}_2$ -FLOWS ON TRACEABLE BRIDGELESS CUBIC GRAPHS.

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ABSTRACT. We investigate nowhere-zero non-conflicting $\mathbb{Z}_2 \times \mathbb{Z}_2$ -flows in bridgeless cubic graphs, following the framework of Mkrтчyan. We show that if such a graph contains a Hamiltonian path (i.e. is traceable) then it admits a perfect matching F such that the contracted graph G/\overline{F} supports a nowhere-zero non-conflicting flow, with the Petersen graph as the sole exception. Our proof relies on the Hamiltonian-path-induced matching and local transformations of perfect matchings, ensuring configurations that force the desired flow. The only unresolved case occurs when the two odd cycles of the 2-factor are linked by a single even component. Thus, our results establish the conjecture for the broad class of traceable cubic graphs while isolating the singular structures where new ideas are needed.

1. INTRODUCTION

The study of flows and edge-colorings in cubic graphs has long been central in graph theory, with deep connections to several enduring conjectures such as those of Jaeger and Berge-Fulkerson. In recent years, Mkrтчyan [Mkr24] introduced and developed the concept of *non-conflicting nowhere-zero $\mathbb{Z}_2 \times \mathbb{Z}_2$ -flows* on cubic graphs, and demonstrated their usefulness for constructing normal edge-colorings and understanding structural properties of such graphs.

Given a finite, undirected, bridgeless cubic graph G , let F be a perfect matching and \overline{F} its complementary 2-factor. Following [Mkr24], a nowhere-zero $\mathbb{Z}_2 \times \mathbb{Z}_2$ -flow θ on the contraction G/\overline{F} is called *non-conflicting* with respect to F if F contains no edge $e = uv$ such that u is incident to an edge with θ -value α and v is incident to an edge with θ -value β , where $\mathbb{Z}_2 \times \mathbb{Z}_2 = \{0, \alpha, \beta, \alpha + \beta\}$.

Mkrтчyan's work established that the existence of a perfect matching F such that G/\overline{F} admits a non-conflicting nowhere-zero $\mathbb{Z}_2 \times \mathbb{Z}_2$ -flow implies that G admits a normal 6-edge-coloring, thus linking the existence of certain flows to classical edge-coloring problems in cubic graphs. Moreover, this framework provides new tools for approaching questions related to the Petersen coloring conjecture, cycle covers, and the structural analysis of non-3-edge-colorable cubic graphs.

Despite these advances, several fundamental questions remain open. In particular, while positive results are known for bipartite cubic graphs, claw-free bridgeless cubic graphs, and graphs admitting a 2-factor with at most two cycles (excluding the Petersen graph), the precise characterization of cubic graphs that admit non-conflicting

flows with respect to all perfect matchings is still incomplete. Notably, the Petersen graph does not admit a non-conflicting nowhere-zero $\mathbb{Z}_2 \times \mathbb{Z}_2$ -flow with respect to any perfect matching, and infinite families of such exceptional graphs have been constructed [Mkr24].

In this paper, we build upon the foundational results of Mkrtchyan and further investigate the interplay between the structure of perfect matchings, 2-factors, and the existence of non-conflicting nowhere-zero $\mathbb{Z}_2 \times \mathbb{Z}_2$ -flows in cubic graphs. Our approach emphasizes the role of Hamiltonian paths and matching transformations, and yields new sufficient conditions for the existence of non-conflicting flows in a broad class of cubic graphs. In particular, we provide constructive techniques for generating perfect matchings that ensure the existence of such flows, and we generalize prior results to cover new families of graphs beyond those previously treated.

The paper is organized as follows. In Section 2, we recall necessary preliminaries and summarize key results from the literature. Section 3 develops the main combinatorial tools for analyzing matchings in graphs with specified cycle structures. Section 4 applies these methods to various families of graphs, including bipartite, homogeneous, and Hamiltonian graphs, establishing new sufficient conditions for the existence of non-conflicting flows. We conclude in Section 5 with a discussion of open problems and directions for further research.

2. PRELIMINARIES

Most of the terminology and definitions in this section are taken from Mkrtchyan [Mkr24]. We work with finite, undirected graphs. When we take contractions we allow loops and parallel edges (i.e., we pass to pseudo-graphs), and a loop contributes 2 to the degree of its incident vertex.

Definition 1 (Graphs and cycles). A *graph* is a pair $G = (V, E)$ with finite vertex set V and edge multiset E . A *cycle* is a connected 2-regular subgraph.

Definition 2 (Degree, cubic, bridgeless). The *degree* of a vertex is the number of incident edge-ends (a loop contributes 2). A graph is *cubic* if every vertex has degree 3. An edge e is a *bridge* if $G - e$ is disconnected; a graph is *bridgeless* if it has no bridge.

Definition 3 (Matching, perfect matching, complementary 2-factor). A *matching* $F \subseteq E(G)$ is a set of pairwise non-incident edges; it is *perfect* if every vertex is incident with exactly one edge of F . In a cubic graph, the complement $\bar{F} := E(G) \setminus F$ is a spanning 2-regular subgraph (a *2-factor*), called the *complementary 2-factor of F* .

Definition 4 (Contraction of a cycle; contraction along a 2-factor). Let C be a cycle of G . The *contraction* G/C is obtained by identifying all vertices of C into a single vertex; edges with both ends in C become loops at that vertex, and edges with exactly one end in C become incident to the new vertex. If \bar{F} is a 2-factor (i.e., a disjoint union of cycles), then G/\bar{F} denotes the graph obtained by contracting *each* cycle $C \subseteq \bar{F}$ to a single vertex. We retain any loops and parallel edges created by these contractions.

Definition 5 ($\mathbb{Z}_2 \times \mathbb{Z}_2$ -flows). Let $\Gamma = \mathbb{Z}_2 \times \mathbb{Z}_2 = \{0, \alpha, \beta, \alpha + \beta\}$. An orientation of G together with a map $\theta : E(G) \rightarrow \Gamma$ is a Γ -flow if at every vertex the sum (in Γ) of values on incoming edges equals the sum on outgoing edges. The flow is *nowhere-zero* if $\theta(e) \neq 0$ for all edges.

Definition 6 (Non-conflicting nowhere-zero $\mathbb{Z}_2 \times \mathbb{Z}_2$ -flow [Mkr24]). Let G be a bridgeless cubic graph, F a perfect matching, and \bar{F} its complementary 2-factor. A nowhere-zero $\mathbb{Z}_2 \times \mathbb{Z}_2$ -flow θ on the contracted graph G/\bar{F} is *non-conflicting with respect to F* if there is no matching edge $e = uv \in F$ such that one endpoint (say u) is incident to an edge of θ -value α while the other endpoint v is incident to an edge of θ -value β . (Here u and v are the vertices of G/\bar{F} corresponding to the cycles of \bar{F} that e meets; if both ends of e lie on the same cycle of \bar{F} , then e becomes a loop after contraction.)

Definition 7 (Hamiltonian path; traceable graph). A *Hamiltonian path* is a path visiting every vertex exactly once. A graph is *traceable* if it contains a Hamiltonian path.

Definition 8 (Bipartite). A graph is *bipartite* if its vertex set can be partitioned as $V(G) = V_1 \cup V_2$ so that every edge has one endpoint in V_1 and the other in V_2 . Equivalently, a graph is bipartite iff it contains no odd cycle.

Lemma 1. The number of vertices on a cubic bridgeless graph G is even.

Proof. The number of edges of G is $|E(G)| = \frac{3n}{2}$ which is an integer if and only if n is even. □

The following lemma was proven by [Mkr24], but we present the proof for the sake of completeness as it is fundamental for many of the subsequent arguments.

Lemma 2 (Mkrtchyan, Observation 1). Let M be any matching of G . If all cycles of \overline{M} are even, then G/\overline{M} admits a nowhere-zero and non-conflicting $\mathbb{Z}_2 \times \mathbb{Z}_2$ -flow.

Proof. Assign $\alpha + \beta$ to every edge of G/\overline{M} . As every vertex of G/\overline{M} has even degree and $2\alpha + 2\beta = 0$, this will be a valid nowhere-zero flow. Because no edge gets an α or β assignment, no conflict can occur. \square

Observation: Any Hamiltonian Graph trivially admits a $\mathbb{Z}_2 \times \mathbb{Z}_2$ -flow

3. MAIN RESULTS

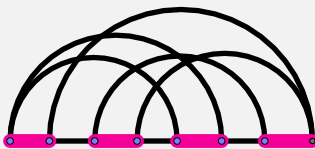
The main result of this paper is stated below.

Theorem 1. For all cubic bridgeless graphs G that admit a Hamiltonian Path there exists a perfect matching F such that G/\overline{F} admits a non-conflicting non-vanishing $\mathbb{Z}_2 \times \mathbb{Z}_2$ -flow with respect to \overline{F} except if G is the Petersen graph.^a

^aI have proven this except for the case where there exists only a singular even component connecting the two odd cycles of the H -matching complement.

In the following let H denote a/the Hamiltonian path of G . Its existence allows us to define an H -matching.

Definition 9 (H-Matching). Starting from one end of H , label the vertices of the Hamiltonian path in order by v_1, \dots, v_n . Define the H -matching M_H of G to be the perfect matching given by the edge set $\{(v_{2k-1}, v_{2k}) : 1 \leq k \leq \frac{n}{2}\}$. One can see that M_H is indeed a matching since every vertex appears at most on one of the edges in M_H . We can also see that M_H is perfect since all vertices of H appear in the matching and thus all vertices of G as H is a Hamiltonian path.



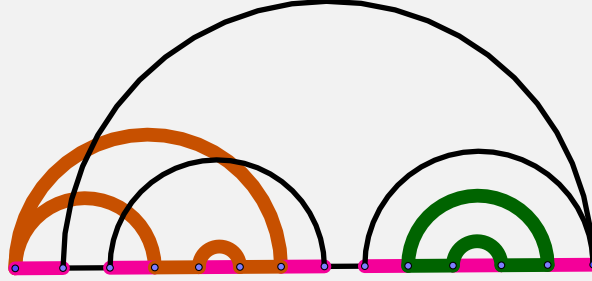
Lemma 3. The number of odd cycles in $\overline{M_H}$ is either 0 or 2. In the latter case, each of the odd cycles contains exactly one of the two endpoints of H .

Proof. Let C be a cycle that does not contain either of the two endpoints of the Hamiltonian path H . Then note that, by construction of the matching M_H , the edges of C lie alternatingly on H and not on H . Because C is a cycle it thus must have an even number of edges.

Suppose now that C is a cycle that contains both endpoints v_1 and v_n of H . Then $|E(\overline{M_H}) \setminus E(C)|$ is even as the complement only contains even cycles. But we

also know that $|E(\overline{M_H})| = \frac{3n}{2} - |E(M_H)| = \frac{3n}{2} - \frac{n}{2} = n$ which is even by Lemma 1. Therefore, C must have even length as well.

Suppose now that v_1 and v_n lie in disjoint cycles C_1 and C_2 . By construction of the matching M_H , we see that C_1 has to contain the two edges adjacent to v_1 that are not in H . Now note that tracing C_1 from v_1 back to v_1 we alternately pass over edges of H and $G \setminus H$. But this time we start and end on an edge of $G \setminus H$. Therefore, C_1 must have odd length. The same argument applies to C_2 . As we have covered all possible cycles in $\overline{M_H}$, the lemma follows.



Because of Lemma 3 we will assume from now on that we encounter two odd cycles in the 2-factor.

A major obstruction for the existence of non-conflicting flows is posed by triangles, but the following lemma will help us avoid them.

Lemma 4. Suppose $\overline{M_H}$ contains triangles. Then we can alter H to another Hamiltonian path H' such that $\overline{M_{H'}}$ does not contain any triangles.

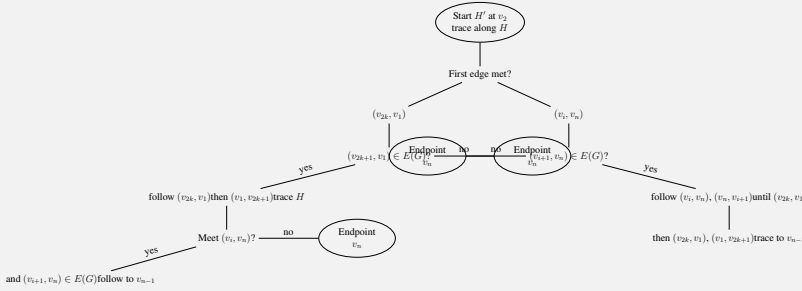
Proof. Lemma 2 implies that the only triangles can occur on the two odd cycles C_1, C_2 containing one of the endpoints v_1, v_2 of H . Thus if C_1 is a triangle, then its edgeset has to be of the form $\{(v_1, v_{2k}), (v_{2k}, v_{2k+1}), (v_{2k+1}, v_1)\}$ for some integer k between 2 and $\frac{n-2}{2}$. k cannot be 1 as otherwise G would contain the bridge (v_3, v_4) . The edge (v_{2k}, v_{2k+1}) lies on H .

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Either we are already triangle free in $\overline{M_H}$ or we can find a Hamiltonian path H' without any triangles using the following algorithm. Without loss of generality we can assume that a triangle appears at v_1 .

- (1) Start the new Hamiltonian path H' at v_2 , trace the Hamiltonian path towards v_n until you meet an edge of one of the following types $(v_{2k}, v_1), (v_i, v_n)$.

- (a) If we first meet (v_{2k}, v_1) , and $(v_{2k+1}, v_1) \in E(G)$, then continue H' via $(v_{2k}, v_1), (v_1, v_{2k+1})$ and then continue along H towards v_n until you either meet a non- H edge of type (v_i, v_n) or the endpoint v_n .
- (i) If we first meet (v_i, v_n) and $(v_{i+1}, v_n) \in E(G)$, then continue the path via $(v_i, v_n), (v_n, v_{i+1})$ and then proceed on H until you meet the new endpoint v_{n-1} .
- (ii) Otherwise v_n is the endpoint of H' as well.
- (b) If we first meet (v_i, v_n) and $(v_{i+1}, v_n) \in E(G)$, then continue the path via $(v_i, v_n), (v_n, v_{i+1})$ until you meet (v_{2k}, v_1) . Then continue H' via $(v_{2k}, v_1), (v_1, v_{2k+1})$ and follow H until you meet the new endpoint v_{n-1} .



Now let us show that the new Hamiltonian path H' will lead to a triangle free $\overline{M}_{H'}$.

Case 1: First suppose that both of the edges coming out of v_n meet H at a vertex to the right of the triangle lying at v_1 . $\overline{M}_{H'}$ will not contain a triangle at its new starting point v_2 . This is because the new odd cycle C'_1 at v_2 will contain the edge (v_2, v_1) as well as the edge (v_1, v_{2k+1}) . But G does not contain the edge (v_{2k+1}, v_2) , since $2k \neq 2$. Thus C'_1 cannot be a triangle.

Similarly, we see that the triangle at v_n will be removed if it exists.

Case 2: Suppose that one of the edges coming out of v_n meets H to the left of the triangle at v_1 and the other edge meets H to the right of the triangle at v_1 . Then our algorithm will again remove the triangle at v_1 but no triangle can appear at v_n since none of the edges coming out of v_n are connected by a single edge.

Suppose both of the non- H edges coming out of v_n lie on the left of the triangle coming out of v_1 . No new triangle can be created: If no triangle exists at v_n in any matching, no other matching can create one. If a triangle exists in the 2-factor, the new path removes it. If a triangle was earlier only not in the 2-factor because of the matching choice, the new Hamiltonian path will change the endpoint to v_{n-1} . A new matching cannot create a triangle at v_{n-1} as that would mean that the triangle is of the form $(v_i, v_{n-1}), (v_{n-1}, v_n), (v_n, v_i)$, but if $i \neq n-2$ we necessarily have that (v_i, v_{n-1}) and (v_n, v_i) are both non- H edges which is impossible as v_i only has one such edge. Therefore, we necessarily have $i =$

$n-2$. But under our new matching $M_{H'}$ the edge (v_{n-2}, v_{n-1}) will be a matching edge. Therefore, this triangle cannot exist in the complement of the matching. Therefore, our complement is triangle free under the new matching. \square

Therefore, we can assume that $\overline{M_H}$ is triangle-free and thus that the two odd cycles, if they exist, each have length at least 5.

Now, we will move on with casework over the nature of the connections between the two odd cycles of the 2-factor. First, we will have to define a few tools.

Definition 10 (Neighboring Edges and Paths). We say that the two edges (e_1, e_2) and (f_1, f_2) are *neighboring* with respect to a perfect matching F if $(e_i, f_j) \in \overline{F}$ for some $i, j \in \{1, 2\}$. We say that two paths P_1 and P_2 are *never neighboring* with respect to a perfect matching F if no matching edge on P_1 neighbors any matching edge on P_2 .

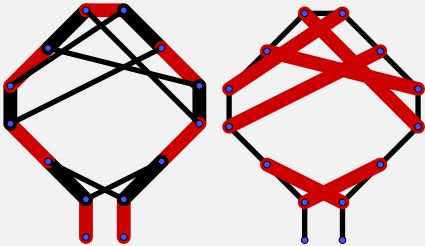
Definition 11 (Even Component). The graph $G \setminus E(C_1 \cup C_2)$ will consist of disjoint connected components. We will case these *even components*. We call an even component *connecting* if connects C_1 and C_2 . We call an even component an *appendix* component if it is only adjacent to one of the odd cycles. Note that each even component E will only contain even cycles of the 2-factor \overline{M} and consist of an even number of vertices. Further, note that an even component can consist of a singular matching edge. We will call an even component *proper* if it is not a singular matching edge.

Lemma 5. Under the matching M_H the Hamiltonian path H has to pass through the union of all connecting components an odd number of times.

Proof. By lemma 2, H has one endpoint on C_1 and the other endpoint on C_2 . Therefore, the lemma follows. \square

3.1. Matching Transformations.

Definition 12 (H -Complement Matching). Let T be a subgraph of G . By the *H -complement matching* we mean the matching defined by the edge set $E(G \setminus H)$.

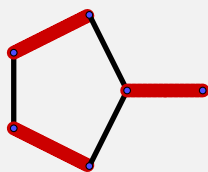


Lemma 6. For an appendix component A the H -complement matching is a matching for all vertices of A except at the leaf nodes connecting A with C_i .

Proof. Every non-leaf node has degree three. Two of the adjacent edges will lie on H . Thus the complement will assign a matching edge to every such vertex. The leaf edges will be left out of the matching as H passes through them. \square

Observation: The H -complement matching adds the hamiltonian path segment to the matching complement. In particular, the two endpoints of the path segment now lie on the same cycle.

Definition 13 (Odd Cycle Matching). Let e be a matching edge (wrt. M_H) leaving one of the odd cycles C_i . By the *Odd Cycle Matching of C_i with respect to e* we mean the matching defined by setting e to be a matching edges as well as alternating edges along C_i starting and ending with the two cycle edges neighboring e on C_i .



Definition 14 (Flipped H Matching). Let S be a segment of H . By the *flipped H matching on S* we mean the matching defined by the edge set $E(H) \setminus M_H$.

Lemma 7. The H -complement matching, the odd cycle matching, and the flipped H matching together combine to a perfect matching of G .

Proof. Each of the three matchings defines a perfect matching on the subgraph that they are acting on. These three subgraphs matchings do not contradict each other since the leaf edges for the H -complement matching and the flipped H matching are never in the new matching. \square

Definition 15 (Standard Matching Transformation). We will call the combination of the H -complement matching, the odd cycle matching, and the flipped H matching the *standard matching transformation*.

Lemma 8. After applying the standard matching transformation, the only two odd cycles can appear at the endpoints of the fixed path segment.

Proof. Note that for all new cycles under the described matching transformations, the edges along them will alternately be matching edges and non-matching edges in the original H -Matching. The only cycles where this can fail are the cycles adjacent to the fixed edge along the original odd cycles. Since the number of vertices in the whole graph is even, the number of odd cycles will have to be odd. Therefore, we again either have 0 or 2 odd cycles. \square

With these tool in hand we can finally start working through the possible cases of connections between the two odd cycles of the 2-factor.

Lemma 9. Suppose there exist 5 Hamiltonian path segments connecting C_1 and C_2 such that each of the segments lies in a different even component. Then a non-vanishing non-conflicting $\mathbb{Z}_2 \times \mathbb{Z}_2$ -flow exists except if G is the Petersen-graph.

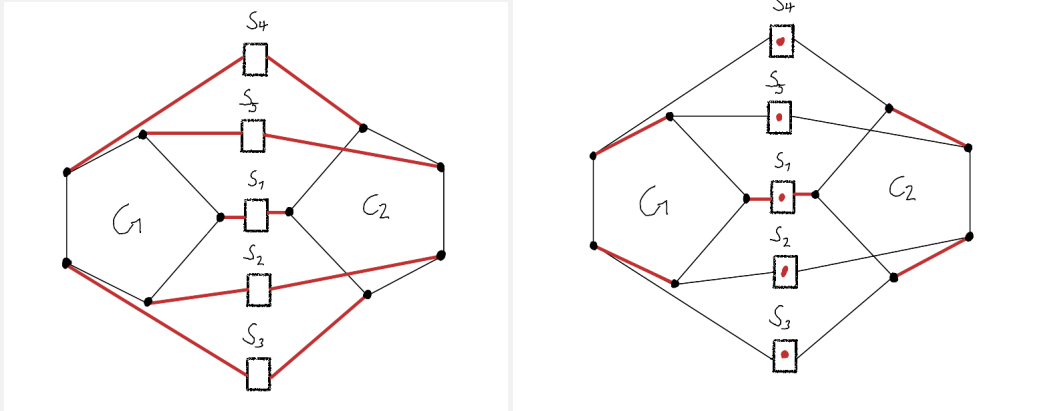
Proof. If we have two segments S_1, S_2 connecting C_1 and C_2 such that S_1 and S_2 are never neighboring, then we are done as we will be able to assign a flow by setting the matching edges along S_1 to α , the matching edges along to S_2 to β and all other matching edges of the graph to $\alpha + \beta$.

Thus, we can assume that every of the five path segments S_1, \dots, S_5 neighbors each of the other four segments on one of the two odd cycles C_1 and C_2 . In particular, note that since G is cubic, this means that every segment will neighbor exactly two of the other segments on each of the two odd cycles. This implies that C_1 and C_2 are both five cycles. In the case that all five segments S_1, \dots, S_5 are singular matching edges, the graph must then be the Petersen-graph.

Therefore, we can assume that at least one of the five segments is not a singular matching edge, but is instead passing through a proper even component E .

Note that in E the complement of the Hamiltonian path H forms a perfect matching, since H occupies exactly 2 out of 3 edges adjacent to any given vertex in E . This fact allows us to perform the following matching transformation. Pick one of the segments S_1 such that one of the segments not in the same component is not just a singular matching edge. This is possible by our earlier assumption. Then, perform the Hamiltonian path segment complement matching transformation to each of the other connecting segments and their respective components. To finally get a matching on the whole graph, set the four edges that are of distance 1 away from S_1 on C_1 and C_2 to be matching edges as well. Then we have a new perfect matching for G . In particular, the cycle structure will now again consist of two odd cycles and potentially a number of even cycles in S_1 . We will again have 5 connecting edges between the two cycles. However, we will not return to the problematic situation that we started with. Instead, one of the odd cycles will now have length at least 6 since one of the connecting segments was not a singular edge. Thus, we cannot have a situation where every matching edge passing from one odd cycle to the other can

be connected to every other such matching edge. Therefore, a non-conflicting non-vanishing flow will be possible on G .



□

Remark. We could generalize this lemma to say: segments are never neighboring inside a given component instead of different components.

Corollary. Suppose there are at least 5 matching edges connecting C_1 and C_2 . Then a non-vanishing non-conflicting $\mathbb{Z}_2 \times \mathbb{Z}_2$ -flow exists if G is not the Petersen graph.

Lemma 10. Suppose there are at least 4 matching edges connecting C_1 and C_2 . Then a nowhere-zero non-conflicting $\mathbb{Z}_2 \times \mathbb{Z}_2$ -flow exists if G is not the Petersen graph.

Proof. By construction of the matching we know that one endpoint of the Hamiltonian path has to lie on C_1 and the other one on C_2 . This implies that there has to be an odd number of Hamiltonian path segments connecting C_1 with C_2 . In particular, this means that there has to exist a 5th Hamiltonian path segment that connects C_1 and C_2 and consequently this lemma follows from the previous one. □

Lemma 11. Suppose there are at least 4 path segments in different components connecting C_1 and C_2 . Then a non-vanishing non-conflicting $\mathbb{Z}_2 \times \mathbb{Z}_2$ -flow exists.

Proof. If we have to never neighboring path segments, we are done. Therefore, we can assume that all segments are neighboring somewhere from now on. As we know that the number of Hamiltonian path segments connecting C_1 and C_2

has to be odd, we know that a fifth segment has to exist. If it lies in a separate component, then we are done by the preceding lemmas. Therefore, we can assume that it lies in the same component as one of our original four segments. Any of the other three segments will still have to neighbor each of the other path segments on one of the two odd cycles. Note that this means in particular that no sixth path segment can exist and further that we have three components with exactly one path segment passing through and one component with two segments passing through. Denote the path segments by s_1, s_2, s_3, s_4, s_5 where s_1 and s_2 lie in the same component. If s_3, s_4, s_5 all neighbor two other segments on C_1 , then C_1 must either be of length five or the path segments must lie in the following order without interruption (up to relabeling within the two classes of segments) on C_1 : $s_1 : s_3 : s_4 : s_5 : s_2$. Consequently, we must have the following order on C_2 : $s_5 : s_1 : s_4 : s_2 : s_3$ which implies that C_2 has length five.

Therefore, we can assume without loss of generality that C_1 is of length 5 and that the path segments lie in the following order on it: $s_1 : s_3 : s_2 : s_4 : s_5$ and that we have the order $s_1 : s_4 : s_3 : s_5 : s_2$ on C_2 . Then we can apply the odd cycle matching to C_1 and C_2 with s_3 being fixed. Apply the Hamiltonian path complement matching to all other even components. Because s_1 and s_2 neighbor s_3 on C_1 and s_4 and s_5 neighbor s_3 on C_2 , we see that the two odd cycles now have at least 4 connecting matching edges of which there exists a pair that does not neighbor or the two cycles are merged. Therefore, we are done. \square

Lemma 12. Suppose there exist at least 3 Hamiltonian path segments in different even components connecting C_1 and C_2 . Then a nonwhere-zero non-conflicting $\mathbb{Z}_2 \times \mathbb{Z}_2$ -flow exists.

Proof. Suppose there are exactly 3 matching edges connecting C_1 and C_2 . Then a non-vanishing non-conflicting $\mathbb{Z}_2 \times \mathbb{Z}_2$ -flow exists.

Denote the three connecting matching edges by e_1, e_2 and e_3 . Once again, note that if e_i and e_j ($i \neq j$) are not neighboring on either of the two odd cycles, then a non-vanishing non-conflicting $\mathbb{Z}_2 \times \mathbb{Z}_2$ -flow can be defined by assigning α to e_i , β to e_j and $\alpha + \beta$ to all other matching edges of G .

Therefore, we can assume that each e_i neighbors each e_j on one of the two odd cycles.

Let us solve this from here with case work.

The 3 endpoints of the e_i lie in a row on each of the two odd cycles. In each of these cases two of the three e_i will neighbor on C_1 as well as on C_2 . Thus, we can merge C_1 with C_2 via a square-matching-edge-flip. Consequently, all cycles in the 2-factor will be even. See the figures below.

Other case: The only other case, up to re-labeling, is that e_1, e_2 and e_3 are intersecting C_1 in that order without any additional vertices in between and that e_1 and e_3 neighbor on C_2 while there are some non zero number of vertices between e_1 and e_2 as well as e_3 and e_2 on C_2 . While this matching configuration will not directly allow a non-vanishing non-conflicting Hamiltonian flow, we will be able to perform a matching transformation that does.

First of all, note that all even components are so called appendix components. They are only adjacent to exactly one of the two odd cycles. Similar to before, we can find a matching on these components by taking the complement of the Hamiltonian path passing through them. To use this, we will have to make sure that the sub-matching on these even component does not contradict the matching of the remaining graph. In order to do this, we add matching edges along the two odd cycles.

For now we assume that there are no additional Hamiltonian path segments connecting C_1 and C_2 . For C_1 , note that the arc connecting e_1 with e_3 which does not intersect e_2 will have an odd number of edges. Thus we can add alternating edges of this arc to the matching such that the edges adjacent to e_1 and e_3 are both in the matching. Then, remove the edges e_1 and e_3 both from the matchin and add the edge connecting e_1 and e_3 on C_2 to the matching. The resulting matching edge collection will indeed be a perfect matching. Furthermore, we claim that this matching will come with a working flow.

To see this, note that all cycles will be of even length. This is true since all appendix cycles will necessarily be even (they are 2-colorable by virtue of the original Hamiltonian path matching), none of the even components attached to C_2 have been altered and the two odd cycles were connected into one.

What if there are other Hamiltonian path segments connecting the two odd cycles as well?

Suppose there are at least two different additional even components connecting the two odd cycles. Then we are in the situation of the 5-lemma.

Therefore, we can assume that there is only one connecting even component E . Similarly to previous arguments we see that there has to be an even number of Hamiltonian path segments going through E .

Suppose there are exactly 2 of them. Then there is only one configuration such that every of the 5 path segments is neighboring each of the other 4 path segments on one of the two odd cycles. See the sketch below for a working matching transformation in order to get a possible flow.

We also need to treat the case where not all path segments are neighboring, but the only such pair is given by the two paths segments going through E . See the only way this can play out in the sketch below. By the given matching edge transformation we will get 5 connecting true edges between the two odd cycles which we handled in the case 5 lemma.

Now, in all other cases we will be able to find a flow since we have a pair of never neighboring path segments connecting C_1 and C_2 .

Now suppose that E actually has at least 4 (remember it cannot be 3 because of parity) Hamiltonian path segments passing through. Then there can exist at most one true connecting edge that neighbors all 4 of these paths. Therefore, we will definitely find a path pair for our flow.

Now suppose that there exists at least 1 path segment that is proper.

We can assume that exactly 3 components exist that connect C_1 and C_2 as the cases with more components are handled by previous lemmas. We will split the proof for this case into three subcases.

All 3 components have exactly one path. Then the same argument has in the 3 matching edge case works, noting that we can force the two ends of a path segment to be connected in the 2-factor by conducting the Hamiltonian path segment complement flip on the said component.

Exactly 1 component has more than one path. Denote the three even connecting components by a, b, c and suppose that c is the one with more than one path. Note that c will have at least 3 Hamiltonian path segments since the number of overall segments has to be odd. Denote the Hamiltonian path segments by $a_1, b_1, c_1, c_2, c_3, \dots$ respectively. Note that if there exist any two path segments that are never neighboring, then we are immediately done. Therefore, we can assume that a_1 has to neighbor b_1 and all of the c_i . This means that if c has more than 3 Hamiltonian path segments, then we are already done. Otherwise we will have the following configuration on C_1 and C_2 (up to relabeling). On C_1 we have $c_1 : a_1 : b_1 : c_2 : \dots : c_3$ and on C_2 we have $c_2 : a_1 : c_3 : b_1 : c_1$. Thus we can perform the matching edge transformation where we do the alternating matching transformation on C_1 and C_2 with a fixed and b, c doing the Hamiltonian path complement flip. Then, under the new matching the two odd cycles will have two non-neighboring connecting matching edges since c is a proper connecting even component.

At least 2 components have more than one path. Suppose all 3 components have more than one Hamiltonian path segment. Then one of the connecting even components has to have at least 3 paths segments. Thus, there exists some Hamiltonian path segment that will have to neighbor at least 5 other path segments in different components. Since it can neighbor at most 4 on the odd cycles, this is impossible. Therefore, we can assume that one even connecting component has exactly one path. Name this component a and its path segment a_1 . Similarly, label the other components b and c and their path segments $b_1, b_2, \dots, c_1, c_2, \dots$. If the Hamiltonian path segment a_1 does not neighbor all of the other connecting path segments, we are done. Therefore, we can assume that there are exactly 2 path segments through each of b and c . Again, we can assume all of these to neighbor all path segments from different components, otherwise we would be done. Thus, we get 2 possible configurations on the odd

cycles. For the first one we get that on C_1 the paths are line up (up to relabeling) as $c_1 : b_1 : a_1 : b_2 : c_2$ and on C_2 as $b_2 : c_1 : a_1 : c_2 : b_1$. The alternating matching edge flip on C_1 and C_2 with a fixed and Hamiltonian path complement flip on b and c will lead to a pair of non-neighboring matching edges between the two new odd cycles since b and c are both proper components.

The other possible configuration is given (up to relabeling) on C_1 as $c_2 : b_1 : a_1 c_1 : b_1$ and on C_2 as $c_1 : b_2 : a_1 : c_2 : b_1$. Performing the alternating matching edge flip on the C_i with a fixed and the Hamiltonian path complement flip on b and c will then again lead to a pair of non-neighboring matching edges between the two new odd cycles.

□

Lemma 13. Suppose there are exactly 2 matching edges connecting C_1 and C_2 . Then a non-vanishing non-conflicting $\mathbb{Z}_2 \times \mathbb{Z}_2$ -flow exists.

Proof. Similarly to the argument in the 4-edge case we see that there will have to exist a third Hamiltonian path segment connecting C_1 with C_2 . Therefore, the case follows from the previous lemma.

□

Lemma 14. Suppose there is exactly 1 matching edge connecting C_1 and C_2 . Then a non-vanishing non-conflicting $\mathbb{Z}_2 \times \mathbb{Z}_2$ -flow exists.

Proof. Because the graph is bridge less, there will have to exist at least one other Hamiltonian path segment connecting C_1 with C_2 . Similarly to previously explained arguments we see that this implies that there are at least 3 Hamiltonian path segments connecting the two odd cycles.

Suppose the latter two path segments are not in the same component. Then the argument from Lemma 18 works again. Suppose thus that the latter two path segments lie in the same component. In particular, suppose that only one proper even connecting component E exists. We then know that the number of path segments through that component has to be even. Suppose we have 6 or more such path segments. Then the singular true connecting edge cannot possible neighbor all 6 paths. Therefore, we will be able to find a non-conflicting flow. Next, assume that we have 4 path segments. If one of them does not neighbor the singular true connecting edge e , then we are done. Therefore, we can assume that two path segments neighbor e on C_1 and the other two neighbor e on C_2 . Then we can perform a matching edge flip by the following algorithm.

- (1) Leave e in the matching.
- (2) Perform the Hamiltonian path complement flip on E and the alternating edge flip along C_1 and C_2 with fixed edge e as well as the respective appendix flips. We will then get a pair of non-neighboring matching edges

connecting the new odd cycles. We can see this the following way. First, denote the four path segments in E by b_1, b_2, b_3, b_4 . Without loss of generality, assume that b_1 and b_2 neighbor e on C_1 and that b_3 and b_4 neighbor e on C_2 . We know that each b_i has to neighbor each other b_j as otherwise, we would have been done in the first place. Suppose b_1 neighbors one of b_3 or b_4 inside of E . Then the Hamiltonian path complement flip will lead to a new matching edge between the two odd cycles along the edge over which b_1 neighbored b_3 or b_4 and this new matching edge will not neighbor e since it is inside of E . It could also be that b_1 neighbors b_3 and b_4 only on the original C_i . Without loss of generality, we can assume that b_1 neighbors b_3 on C_1 and b_4 on C_2 . Then the two edges that caused the neighboring will turn into matching edges between the two new odd cycles. Because b_1 is a proper path segment (otherwise it would not have been in E), these two new connecting matching edges are not neighboring and we are done.

Consequently, we can assume that there exist only two path segments in E . If one of them does not neighbor e , then we are done. Therefore, we can assume that either both path segments neighbor e at the same odd cycle or on opposing cycles.

Suppose that both of them neighbor e on the same cycle. Without loss of generality, assume that they do so on C_1 . Then we can perform the following matching edge flip.

□

Lemma 15. Suppose there are no matching edges connecting C_1 and C_2 . Then a non-vanishing non-conflicting $\mathbb{Z}_2 \times \mathbb{Z}_2$ -flow exists.

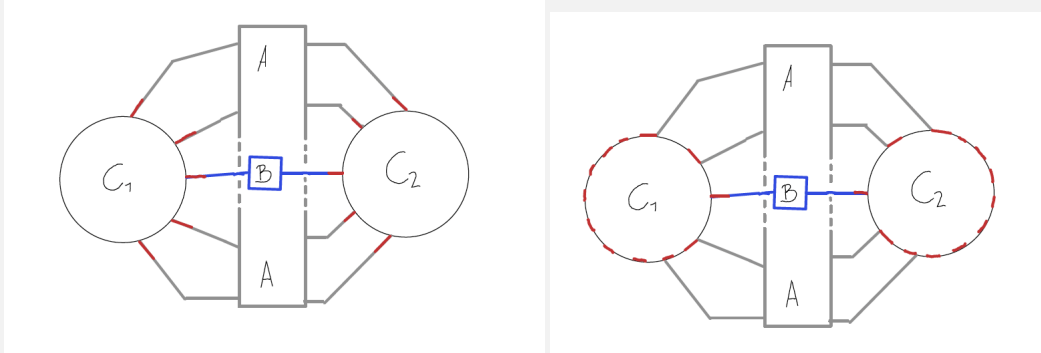
Proof. Case 3+ components If we have 3 or more connecting components between the two odd cycles, see the generalized 3 edge case.

Case 2 components If we have 2 connecting components and one of them has at least 5 path segments, then we are done as well as a path segment from component A can neighbor at most 4 path segments from component B since G is cubic.

Subcase 4:3

Subcase 4:1 On component A perform the Hamiltonian path complement matching edge flip. In order to extend this to a matching edge flip for the whole graph, we also do the alternating matching edge flip along C_1 and C_2 where the two endpoints of B are our respective fixed points. Since the only odd cycles in the new matching complement can come from the cycles containing the endpoints of B , we either have 0 odd cycles (in which case the two endpoints lie on the same cycle) or 2 odd cycles. In the latter case we either received a new matching edge connecting the two odd cycles inside A or there exist two paths of A that never neighbor. In the first case, we will have a non conflicting flow by assigning α to the edges along the path segment in B and β to the new

connecting matching edge. In the second case, we will have a non-conflicting flow on the original matching by assigning α and β respectively to the two path segments in A that never neighbor.



Subcase 3:2 Perform the following matching edge transformation. On B , perform the connecting operation and on A perform the path segment complement operation. On C_1 and C_2 perform the alternating matching edge operation with the one fixed path segment of B providing the fixed points. This will evidently result in a new matching. Again, the only odd cycles in the new matching can appear at the endpoints of the fixed path segment f in B . Suppose there exists a path segment in A that never neighbors f . Then we would have been done in the original matching already. Therefore, we can assume that all 3 of them are adjacent to f on one of the two odd cycles. Now again, we either have that the updated matching gives us an additional edge in B between the new odd cycles which does not neighbor f or otherwise we would have had non-neighboring path segments in the original matching already.

Subcase 2:1 Should one of the 2 path segments in A not neighbor B anywhere, we would be done. Therefore, we have two (not necessarily mutually exclusive) possible cases. Either, both segments of A neighbor B on C_1 or one neighbors B on C_1 and the other neighbors B on C_2 .

In the first case we can perform the following matching edge transformation: Do the Hamiltonian path complement flip on B and the 1 segment connecting operation on A . Do the alternating edge flip on C_2 . Only add a matching edge between the segment of A that does not end in matching edges anymore and B . Then the only possible sources of odd cycles are the C_2 endpoint of the unchanged path segment in A and the cycle now containing C_1 . By construction we would now have at least one true connecting edge between the two odd cycles (end-edge of unchanged segment of A).

In the second case, we can basically proceed as in the previous splits. Perform the matching transformation given by the Hamiltonian path complement flip on A and the alternating edge operation on C_1 with fixed points being the endpoints of. We will then have two non-neighboring connecting segments between the two odd cycles as we introduced a new matching edge or we will have found that the two original path segments never neighbored (not sure that is possible in the 2-segment case).

Case 1 component This case remains unsolved. □

4. GRAPHS FOR WHICH EVERY MATCHING ALLOWS A FITTING FLOW

Another interesting problem is to ask for which graphs any perfect matching will allow a non-conflicting nowhere-zero $\mathbb{Z}_2 \times \mathbb{Z}_2$ -flow. Such graphs will be useful tools for showing the existence of fitting flows on other graphs.

Remark. Recall that a sufficient condition for a perfect matching M to admit a non-vanishing non-conflicting $\mathbb{Z}_2 \times \mathbb{Z}_2$ on the contraction G/\overline{M} with respect to \overline{M} is that all cycles in \overline{M} are even.

Proof. In such a case we can just assign $\alpha + \beta$ to every matching edge. □

Therefore, we get the following simple corollary.

Corollary. Every bipartite cubic bridgeless graph G admits a non-conflicting non-vanishing $\mathbb{Z}_2 \times \mathbb{Z}_2$ flow on G/\overline{F} with respect to \overline{F} for any perfect matching F .

However, our sufficient condition did not require that all cycles are even, only that all cycles in the complement of any given matching are even. Do non-bipartite graphs exist that satisfy this condition? The answer, given by Abreu and colleagues [Abr+], is positive.

I have also found another family of graphs for which every matching admits a flow.

Example (n -Gon Depth Graph (odd n)). Define the n -Gon Depth Graph GD_n to be the graph on $2n$ vertices with vertex set $V(GD_n) := \{u_i, v_i : i \in [1, \dots, n]\}$ and edge set

$$E(GD_n) := \{u_i u_{i+1}, v_i v_{i+1} : i \in [1, \dots, n]\} \cup \{u_i v_i : i \in [1, \dots, n]\}$$

where we define $u_{n+1} := u_1$ and $v_{n+1} := v_1$.

Proof. We claim that the only perfect matching M for which \overline{M} does contain odd cycles is the one given by $\{u_i v_i : i \in [1, \dots, n]\}$. This is because of symmetry arguments.

We claim that $u_i u_{i+1}$ can be a matching edge if and only if $v_i v_{i+1}$ is one as well. First note that an odd number of $u_i v_i$ edges has to be in the matching. Then the choice of $u_i v_i$ edges already uniquely forces the choice of matching edges along the U and the V cycle.

This means in particular that any cycle has to be symmetric and thus a working flow has to exist. □

Remark. Note that for odd n the graph GD_n does not have a perfect matching whose complement consists of odd cycles. Therefore, GD_n does not satisfy the E2F property for odd n .

5. CONCLUSION AND OUTLOOK

In this work we extended the framework of Mkrtchyan by showing that every traceable bridgeless cubic graph, namely, every cubic graph that admits a Hamiltonian path, supports a perfect matching F such that the contracted graph G/\overline{F} admits a nowhere-zero non-conflicting $\mathbb{Z}_2 \times \mathbb{Z}_2$ -flow, with the notable exception of the Petersen graph. Our approach relies on the perfect matching induced by the Hamiltonian path and on a catalogue of local transformations of matchings that either merge the two odd cycles of \overline{F} or produce two non-neighbouring matching edge paths, both of which guarantee the existence of a non-conflicting flow.

The one subcase that our arguments do not resolve occurs when the Hamiltonian-path-induced matching yields a 2-factor whose two odd cycles are linked solely by a single connected even component. In this degenerate configuration our local modifications fail, and we cannot currently guarantee the existence of a non-conflicting flow. Further progress will likely require a deeper understanding of these singular linking components.

Mkrtchyan emphasized the role of hypohamiltonian graphs (non-Hamiltonian graphs where every vertex deletion yields a Hamiltonian graph) as a potential source of counterexamples to flow conjectures. Our results show that in the presence of a Hamiltonian path, the non-conflicting flow property holds broadly, except in the unresolved subcase above. Thus, any further obstructions must either arise from these delicate single-connector configurations or else from genuinely non-traceable graphs, with hypohamiltonian bridgeless cubic graphs, led by the Petersen graph, standing out as natural candidates.

Future directions. Several natural avenues remain open:

- *Resolving the singular-connector case.* Can one show that even when a single even component connects the two odd cycles of the 2-factor, a nowhere-zero non-conflicting flow still exists? Or do there exist counterexamples in this regime?
- *Hypohamiltonian graphs.* Beyond the Petersen graph, are there other hypohamiltonian bridgeless cubic graphs that fail to admit non-conflicting flows under every perfect matching?
- *Every-matching graphs.* While we demonstrated existence for at least one perfect matching, classifying graphs for which every perfect matching works (E2F graphs) remains open.

- *Algorithmic and structural extensions.* Efficiently detecting the “bad” singular-connector subcase and adapting the techniques to other flow groups ($\mathbb{Z}_k \times \mathbb{Z}_k$) could broaden the applicability of these methods.

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